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IPA Derivatives for Make-to-Stock Production-Inventory Systems with Backorders

Yao Zhao · Benjamin Melamed

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Abstract A single-stage Make-to-Stock (MTS) production-inventory system consists of a production facility coupled to an inventory facility, and is subject to a policy that aims to maintain a prescribed inventory level (called *base stock*) by modulating production capacity. This paper considers a class of single-stage, single-product MTS systems with backorders, driven by random demand and production capacity, and subject to a continuous-review base-stock policy. A model from this class is formulated as a stochastic fluid model (SFM), where all flows are described by stochastic rate processes with piecewise- constant sample paths, subject to very mild regularity assumptions that merely preclude accumulation points of jumps with probability 1. Other than that, the MTS model in SFM setting is *nonparametric* in that it assumes no specific form for the underlying probability law, and as such is quite general. The paper proceeds to derive formulas for the (stochastic) IPA (Infinitesimal Perturbation Analysis) derivatives of the sample-path time averages of the inventory level and backorders level with respect to the base-stock level and a parameter of the production rate. These formulas are comprehensive in that they are exhibited for any initial condition of the system, and include right and left derivatives (when they do not coincide). The derivatives derived are then shown to be unbiased and their formulas are seen to be amenable to fast computation. The generality of the model and comprehensiveness of the IPA derivative formulas hold out the promise of gradient-based applications. More specifically, since the base-stock level and production rate are the key control parameters of MTS systems, the results provide the theoretical underpinnings for optimizing the design of MTS systems and for devising prospective on-line adaptive control algorithms that employ IPA derivatives. The paper concludes with a discussion of those issues.

Y. Zhao (✉) · B. Melamed
Department of MSIS, Rutgers Business School – Newark and New Brunswick,
Rutgers University, 94 Rockafeller Rd., Piscataway, NJ 08854, USA
e-mail: yaozhao@andromeda.rutgers.edu

B. Melamed
e-mail: melamed@rbs.rutgers.edu

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1 Introduction

Production-inventory systems consist of production facilities that feed replenishment product to inventory facilities, driven by random demand and possibly random production processes, as well as feedback information from inventory to production facilities. In simple MTS systems, a *stage* is comprised of a single-product replenishment flow between a pair of coupled production-inventory facilities, such that production is modulated by inventory state information. An important instance of production-inventory systems is the *Make-to-Stock (MTS)* class, where the inventory facility sends its state information to the production facility as a control signal, which modulates production with the aim of maintaining the inventory level at a prescribed level, called *base stock level*. Such systems can admit backorders when stock is depleted, or suffer lost sales. This paper is concerned with MTS systems with backorders (see Section 2), while a forthcoming paper will treat MTS systems with lost sales.

Economic considerations in supply chains call for effective control of inventory levels and production rates, in order to optimize some prescribed performance metric. This motivates on-line algorithms that can adaptively control such systems over time with the objective of minimizing the inventory on-hand without compromising customer service metrics. To this end, we propose to use *IPA* (*Infinitesimal Perturbation Analysis*) derivatives of selected random variables [for comprehensive discussions of IPA derivatives and their applications, refer to Glasserman (1991), Ho and Cao (1991) and Fu (1994a, 1994b)]. IPA derivatives provide sensitivity information on system metrics with respect to control parameters of interest, and as such can serve as the theoretical underpinnings for on-line control algorithms. Specifically, let $L(\theta)$ be a random variable, parameterized by a generic real-valued parameter θ chosen from a closed and bounded set Θ. The IPA derivative of $L(\theta)$ with respect to θ is the random variable $\frac{d}{d\theta}L(\theta)$, provided that it exists almost surely. An IPA derivative is said to be *unbiased*, if the expectation and differentiation operators commute, namely, $E[\frac{d}{d\theta}L(\theta)] = \frac{d}{d\theta}E[L(\theta)]$; otherwise, it is said to be *biased*. Sufficient conditions for unbiased IPA derivatives are given in the following result.

Fact 1 (see Rubinstein and Shapiro (1993), Lemma A2, p. 70)
An IPA derivative $\frac{d}{d\theta}L(\theta)$ is unbiased, if

(a) For each $\theta \in \Theta$, the IPA derivatives $\frac{d}{d\theta}L(\theta)$ exist w.p.1 (with probability 1).
(b) W.p.1, $L(\theta)$ is Lipschitz continuous in Θ, and the (random) Lipschitz constant has a finite first moment.

For IPA-based applications to be general and efficacious, it is necessary that the requisite IPA derivative formulas satisfy the following requirements:

1. For usability, they should be *comprehensive* in the sense that they are valid for any initial condition of the system. In particular, if a left-derivative does not coincide with its right-derivative counterpart, then both should be exhibited.

2. For statistical accuracy, they should be *unbiased*.
3. For generality, they should be *nonparametric* in the sense that they are solely computable from the sample path observed without making any distributional assumptions on the underlying probability law.
4. To enable on-line applications, they should be *fast* to compute.

Most papers on production-inventory systems (and MTS systems in particular) postulate specific probability laws that govern the underlying stochastic processes (e.g., Poisson demand arrivals and exponential service times). For simple systems, such as the one-stage MTS variety, closed-form formulas of key performance metrics (e.g., statistics of inventory levels and lost sales or backorders) have been derived as functions of control parameters. For example, Zipkin (1986) and Karmarkar (1987) obtain the optimal control of these systems with respect to the batch sizes and re-order points by standard optimization techniques. For more complex MTS systems, such as the multi-stage serial variety, closed-form formulas are not available. A sample path analysis is carried out by Buzacott et al. (1991) for a 2-stage production system which is governed by the continuous-time base-stock policy. Diffusion models and deterministic fluid models have been proposed in order to mitigate the analytical and computational complexity of performance evaluation and optimal control. For example, Wein (1992) used a diffusion process to model a multi-product, single-server MTS system, while Veatch (2002) discussed diffusion and fluid-flow models of serial MTS systems. Note, however, that diffusion models require a heavy traffic condition in order to be valid approximations (Wein, 1992). In a similar vein, while deterministic fluid-flow models provide valuable insights into the control rules of such systems, deterministic modeling may well result in substantial numerical errors (Veatch, 2002).

Simulation has been widely used to study the performances of complex production-inventory systems under uncertainty. Glasserman and Tayur (1995) considered a class of production-inventory systems under the so-called periodic-review, modified base-stock policy, and estimated its performance metrics and IPA derivatives using simulation. While periodic-review policies evaluate system performance at discrete review times, discrete-event simulation, in contrast, can track system performance continuously, but this can be overly time consuming for large-scale systems, due to the large number of events that need to be processed (e.g., arrivals and service completions). All in all, most papers on stochastic production-inventory systems postulate a specific underlying probability law, and focus on off-line control and optimization algorithms.

Recent work has sought to address these shortcomings in the context of fluid-flow queueing systems, and especially, the *stochastic fluid model (SFM)* setting, where transactions carry fluid workload, random discrete arrivals become random arrival rates and random discrete services become random service rates. SFM-like settings represent an alternative (continuous or fluid-flow) queueing paradigm, which differs from the traditional (discrete) queueing paradigm in the way workload is transported in the system.[1] Both paradigms are set in a network of nodes, each of which houses a server and a buffer, where network sources and sinks are viewed as

[1] For simplicity we address only open networks in this discussion.

exogenous nodes, and all others as endogenous nodes. Transactions representing parcels of workload arrive at the network from some source, traverse the network according to some itinerary, and then depart the network at some sink. The two queueing paradigms differ, however, in the way workload moves in the system. In the discrete queueing paradigm, transaction workload moves "abruptly" among nodes following a service time, while in the continuous queueing paradigm, transaction workload moves "gradually" (i.e., flows like fluid) for the duration of its service time.

A heuristic modeling rationale underlying SFM systems is the assumption that individual transactions carry miniscule workload as compared to the entire transaction flow, so the effect of individual transactions is infinitesimal and akin to "molecules" in a fluid flow. Furthermore, in many cases, a transaction workload *does* move gradually from one node to another, rather than abruptly (e.g., a conveyor belt carrying bulk material, loading and unloading a truck, train, etc.) In fact, discrete queueing systems can be abstracted as "limiting cases" of continuous queueing systems, where the flow rate is zero when a transaction is still, but at the moment of motion the flow rate becomes momentarily infinite; in other words, the flow rate is akin to a Dirac function. Pursuing this line of reasoning, the "Dirac pulses" of flow rates in a discrete queueing system can be approximated by high flow rates of short duration in a continuous queueing system. Whichever reasoning is used, the modeler can often choose to model a queueing system using either paradigm on equal footing. Finally, we point out that ceteris paribus, SFM systems enjoy an important advantage over their discrete counterparts: IPA derivatives in SFM setting are *unbiased*, while their counterparts in discrete queueing systems are by and large *biased* (Heidelberger et al., 1988). Thus, the local shape of sample paths in the fluid-flow paradigm confers technical advantages on them. IPA derivatives, derived in SFM setting, can provide important information and insights for their *discrete counterparts*, by applying derivative formulas obtained in SFM setting to queueing systems that have been traditionally viewed as belonging to the discrete queueing paradigm. While preliminary unpublished work by one of the authors suggests that this approach is viable, more work is needed to establish its broad applicability.

Motivated by the considerations above, Wardi et al. (2002) derived IPA derivatives in SFM setting; we henceforth refer to this approach as *IPA-over-SFM*. The paper considered two performance metrics: loss volume and buffer-workload time average; each of these metrics was differentiated with respect to buffer size, a parameter of the arrival rate process and a parameter of the service rate process. The paper showed the IPA derivatives to be unbiased, easily computable and non-parametric. Consequently, these derivatives can be computed in simulations, or in the field, and the values can have potential applications to on-line control and stochastic optimization. Paschalidis et al. (2004) treated multi-stage MTS production-inventory systems with backorders in SFM setting. Assuming that inventory at each stage is controlled by a continuous-time base-stock policy, the paper computed the right IPA derivatives of the time averaged inventory level and service level with respect to base-stock levels, and used them to determine optimal base-stock levels at each stage. Zhao and Melamed (2004) applied the IPA-over-SFM approach to a class of single-product, single-stage MTS systems with backorders. Using a different proof methodology from that of Paschalidis et al. (2004), this paper derived the IPA formulas of the time averaged inventory level and backorder with respect to the base-stock level, as well as a parameter of the production rate process. The goal of this paper is to derive IPA derivatives for Make-to-Stock systems with backorders, and to show them to be unbiased. The key contributions of this paper are two-fold.

The first contribution is the derivation of IPA derivative formulas with respect to the base-stock level for *all* initial inventory states, including those that lie *above* the base-stock level. In contrast, the above-cited references consider only a subset of initial inventory states; for example, Zhao and Melamed (2004) restricts such systems to start with a base-stock level of inventory, while Paschalidis et al. (2004) considers initial inventory states that lie only below the base-stock level. In fact, we show in this paper that transient IPA derivatives depend strongly on the initial inventory state, and in some cases, only sided IPA derivatives exist. The importance of our contribution stems from potential applications of IPA derivatives to on-line control of MTS systems. Clearly, on-line control applications mandate the computation of IPA derivatives for *all* initial inventory states, as well as all sided derivatives, since a control action can change system parameters at a variety of system states (which are then considered as new initial states). Moreover, it obviously makes little or no sense to wait for the system to return to selected inventory states for which IPA derivatives are known, as this could suspend control actions over extended periods of time. For example, consider the situation where an IPA-based control action sets the base-stock level to coincide with the current inventory level (this could happen in applications where inventory levels are discrete), in which case this paper shows that the sided derivatives exist but are not equal. These sided derivatives would be needed in due time to decide on the next control action, where a base-stock level lowering action would call for left IPA derivatives, while a base-stock level raising action would call for right IPA derivatives; note that the inventory level just after each control action is considered to be the new initial inventory state for the purpose of computing the new IPA derivatives. We point out that Zhao and Melamed (2004) also considers IPA derivatives from an initial inventory state that coincides with the base-stock level, but the initial inventory state and base-stock level are required there to vary together, which simplifies the analysis, but does not admit on-line control applications.

The second contribution of this paper is the derivation of IPA derivative formulas with respect to a production-rate parameter, which models the production capacity that replenishes the inventory system. Here, our results generalize Wardi et al. (2002) and Zhao and Melamed (2004), which only consider the case where the left and right IPA derivatives coincide. In contrast, this paper drops this restriction and derives all sided IPA derivative formulas, thereby extending the applicability of IPA-based on-line control.

The computation of the general IPA derivatives in this paper requires major extensions of the results in the open literature, culminating in more elaborate formulas. We show that long-run IPA derivatives with respect to the base-stock level parameter are simpler, and in fact, coincide with published results for a subset of initial inventory states (e.g., Paschalidis et al. (2004), Zhao and Melamed (2004)). However, as noted above, IPA-based on-line control applications cannot rely on long-run IPA derivatives, but must generally utilize their transient counterparts. We mention that while this paper focuses on the IPA derivatives in MTS systems, our ultimate goal is to use derivative information for on-line control and optimization of supply chains, which will be the subject of further research.

Throughout the paper, we use the following notational conventions and terminology. The indicator function of set A is denoted by 1_A and $x^+ = \max\{x, 0\}$, whereas $f(x+)$ and $f(x-)$ denote the right and left limits of f at x and $\frac{d}{dx^+} f(x)$ and $\frac{d}{dx^-} f(x)$ denote, respectively, the right and left derivatives of f at x. A function $f(x)$ is said to be *locally differentiable at* x if it is differentiable in a neighborhood of x; it is said to be *locally independent of* x if it is constant in a neighborhood of x.

The rest of the paper is organized as follows. Section 2 presents the production-inventory model under study. Section 3 provides variational bounds for system metrics. Section 4 derives IPA derivative formulas and shows them to be unbiased. Finally, Section 5 discusses the results, their significance and their use in prospective design and control applications.

2 The Make-to-Stock Model

Consider the traditional single-stage, single-product MTS system, consisting of a production facility and an inventory facility. The two facilities interact: the latter sends back orders to the former, while the former produces stock to replenish the latter. The production facility is comprised of a queue that houses a production server (a single machine, a group of machines or a production line), preceded by an infinite buffer that holds incoming production orders. We assume that the production facility has an unlimited supply of raw material, so it never starves. The inventory facility satisfies incoming demands on a first come first serve (FCFS) basis, and is controlled by a continuous-time base-stock policy with some base-stock level $S > 0$. More specifically, the inventory and production facilities are coupled, and have two operational modes as follows:

Normal mode. While the inventory level does not exceed S, the inventory facility places the orders of incoming demands as discrete production jobs in the production facility's buffer according to some operational rule (to be detailed below). The production facility fills these outstanding orders and replenishes the inventory facility back to its base-stock level, but no higher. We also refer to this operational mode as *normal operation*, because the system strives to reach an inventory level S, and in so doing, it maintains an inventory level not exceeding S.

Overage mode. While the inventory level exceeds S (this could happen, for example, as a result of a control action that lowered S), the production facility buffer is empty, so production is temporarily suspended until the inventory level reaches or crosses S from above, at which point normal operation is resumed. We also refer to this operational mode as *overage operation*.

The demand process consists of an interarrival-time process of demands and their random magnitude. Demands arrive at the inventory facility and are satisfied from inventory on hand (if available). Otherwise, when an inventory shortage is encountered, the behavior of the MTS queue is governed by the *backorder rule* as follows: Any shortage of inventory is backordered from the production facility, and the demand waits in a FCFS buffer at the inventory facility until the production facility replenishes the inventory facility with the shortage amount. Thus, the system's overall actions aim to move the inventory level to the base-stock level, S.

2.1 Mapping MTS Systems to SFM Versions

We next proceed to map the traditional discrete MTS system with backorders into an SFM version, as depicted in Fig. 1. Level-related stochastic processes are mapped into fluid versions of their traditional counterparts in a natural way, as follows:

Inventory level. The traditional jump process of the level of inventory on hand at the inventory facility is mapped to a fluid-level counterpart, $\{I(t)\}$, where $I(t)$ is the (fluid) volume of inventory on-hand at time t.

Backorders level. The traditional jump process of the level of backorders at the inventory facility is mapped to a fluid-level counterpart, $\{B(t)\}$, where $B(t)$ is the (fluid) volume of backorders at time t.

Outstanding orders. The traditional jump process of the level of outstanding orders in the buffer of the production facility is mapped to a fluid-level counterpart, $\{X(t)\}$, where $X(t)$ is the (fluid) volume of outstanding orders at time t.

Traffic-related stochastic processes in Fig. 1 are mapped into fluid versions of their traditional counterparts, as follows:

Arrival rate. The traditional arrival process of discrete demands at the inventory facility is mapped to a fluid-flow stochastic process, $\{\alpha(t)\}$, where $\alpha(t)$ is the rate of incoming demands at time t.

Production rate. The traditional service (production) process of discrete product at the production facility is mapped to a fluid-flow stochastic process, $\{\mu(t)\}$, where $\mu(t)$ is the production rate at time t.

Outstanding order rate. The traditional arrival process of signals for placing discrete outstanding orders at the production facility is mapped to a fluid-flow stochastic process, $\{\lambda(t)\}$, where $\lambda(t)$ is the rate of incoming outstanding orders at time t.

Replenishment rate. The traditional replenishment process of discrete replenished product from the production facility to the inventory facility is mapped to a fluid-flow stochastic process, $\{\rho(t)\}$, where $\rho(t)$ is the replenishment rate of product at time t.

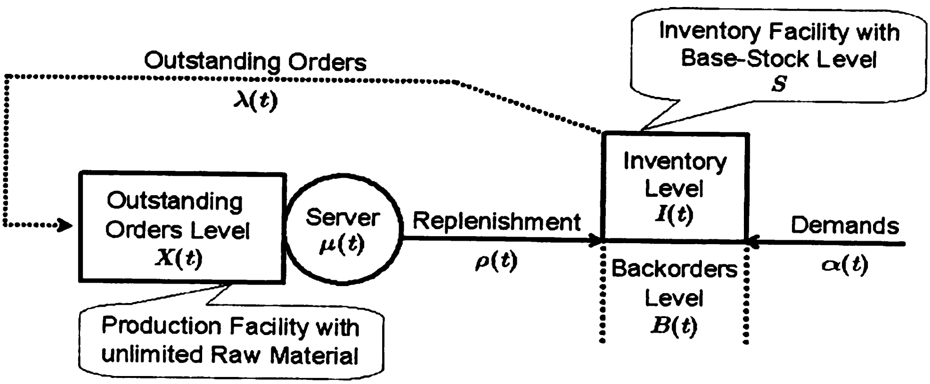

Fig. 1 The Make-to-Stock production-inventory system with backorders

We now proceed to exhibit the formal definitions of all fluid-model components of the MTS system with backorders.

During overage operation, the inventory process is governed by the one-side stochastic differential equation

$$\frac{d}{dt^+} I(t) = -\alpha(t), \tag{2.1}$$

and

$$B(t)=0, \quad X(t)=0 \tag{2.2}$$

$$\lambda(t)=0, \quad \rho(t)=0. \tag{2.3}$$

During normal operation, the model satisfies the conservation relation,

$$X(t) + I(t) - B(t) = S, \tag{2.4}$$

where

$$I(t) = [S - X(t)]^+, \tag{2.5}$$

$$B(t) = [X(t) - S]^+, \tag{2.6}$$

and the outstanding orders process is governed by the sided stochastic differential equation,

$$\frac{d}{dt^+} X(t) = \begin{cases} 0, & \text{if} \quad X(t) = 0 \ \text{ and } \ \alpha(t) \le \mu(t), \\ \alpha(t) - \mu(t), & \text{otherwise}, \end{cases} \tag{2.7}$$

The arrival-rate process of outstanding orders is given by

$$\lambda(t) = \begin{cases} 0, & \text{if} \ \ I(t) > S \\ \alpha(t), & \text{if} \ \ I(t) \le S. \end{cases} \tag{2.8}$$

and the replenishment-rate process is given by

$$\rho(t) = \begin{cases} \mu(t), & \text{if } X(t) > 0 \\ \min\{\mu(t), \lambda(t)\}, & \text{if } X(t) = 0. \end{cases} \tag{2.9}$$

2.2 Performance Metrics and Parameters

Let $[0, T]$ be a given finite time interval, during which system performances are evaluated before a control action regarding the inventory policy and/or production rate is taken. One should not confuse T with the review period of a periodic-review inventory policy.

In this paper, we will be interested in the following random variables, to be henceforth referred to as *performance metrics*.

Inventory time average. The time average of fluid volume of inventory on-hand over the interval $[0, T]$, given by

$$L_I(T) = \frac{1}{T}\int_0^T I(t)\,dt. \tag{2.10}$$

Backorder time average. The time average of fluid volume of backorders over the interval $[0, T]$, given by

$$L_B(T) = \frac{1}{T}\int_0^T B(t)\,dt. \tag{2.11}$$

Observe that the metrics $L_I(T)$ and $L_B(T)$ are random variables for each T.

Let $\theta \in \Theta$ denote a generic parameter of interest with a closed and bounded domain Θ. We write $S(\theta)$, $\mu(t,\theta)$, $L_I(T,\theta), L_B(T,\theta)$ and so on to explicitly display the dependence of a performance random variable on its parameter of interest. Our objective is to derive formulas for the IPA derivatives $\frac{d}{d\theta}L_I(T,\theta)$ and $\frac{d}{d\theta}L_B(T,\theta)$ in the SFM setting, using sample path analysis, and to show them to be unbiased.

The parameters of interest in this section are listed below:

Base-stock level. The base-stock level of the inventory facility,

$$S(\theta) = \theta, \quad \theta \in \Theta. \tag{2.12}$$

Production rate parameter. A parameter of the production rate process, such that

$$\frac{d}{d\theta}\,\mu(t,\theta) = 1, \quad t \in [0,T], \quad \theta \in \Theta, \tag{2.13}$$

interpreted as an additive scaling parameter of the production rate.

2.3 Assumptions

The notion of *sample path events* pertains to a property of a time point along a sample path (not to be confused with the ordinary notion of events as aggregates of sample paths); the distinction can be discerned by context. Similarly to Wardi et al. (2002), we define two types of sample path events:

Exogenous events. An *exogenous event* occurs whenever a jump occurs in the sample path of $\{\alpha(t)\}$ or $\{\mu(t)\}$.

Endogenous events. An *endogenous event* occurs whenever a time interval is inaugurated, in which $X(t) = 0$ or $X(t) = S$.

Throughout this paper, we assume the following mild regularity conditions [cf. Wardi et al. (2002)].

Assumption 1

(*a*) *The demand rate process, $\{\alpha(t)\}$, and the production rate process, $\{\mu(t)\}$, have right-continuous sample paths that are piecewise-constant w.p.1.*

(*b*) *Each of the processes, $\{\alpha(t)\}$ and $\{\mu(t)\}$, has a finite number of discontinuities in any finite time interval w.p.1, and the time points at which the discontinuities occur are independent of the parameters of interest.*
(*c*) *No multiple events occur simultaneously w.p.1.*

While parts (a) and (c) above are mild regularity assumptions, part (b) merits additional motivation as follows. It makes sense to model the demand arrival-rate process, $\{\alpha(t)\}$, as exogenous to the system, and as such we assume it to be independent of any parameter of interest, and this is true in particular of its discontinuity points. The production-rate process, $\{\mu(t)\}$, may depend on a scaling parameter, θ (see equation (2.13)), but its discontinuity points are assumed independent of θ. Note that such discontinuities model a change in production capacity which do not depend on scaling of the production rate.

The following observations follow from Assumption 1.

OSERVATION 1

1. *W.p.1, there exists a finite integer $N \geq 0$ and a sequence of (random) time points $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T$, such that the process $\{\alpha(t) - \mu(t)\}$ is constant over each interval (T_n, T_{n+1}), $n = 0, \cdots, N$, and each time point T_n, $1 \leq n \leq N$, is a jump point of the process.*
2. *The process $\{\alpha(t) - \mu(t)\}$ is constant over each time interval (T_n, T_{n+1}), $n = 0, \cdots, N$.*

Proof: To prove the first observation, note that the finiteness of N follows from part (b) of Assumption 1, while the strict inequalities are a consequence of part (c) of Assumption 1. The second observation follows directly from the first one. ■

Finally, we shall be interested in pairs of systems, the original system (indexed by θ) and a perturbed system (indexed by $\theta \pm \Delta\theta$), both starting at the same initial conditions. To simplify the notation in the sequel, we shall also make the following assumption, without any loss of practical generality.

ASSUMPTION 2 *The initial inventory level and initial backorder level do not depend on θ, namely, $I(0, \theta) = I(0)$ and $B(0, \theta) = B(0)$, for all $\theta \in \Theta$.*

3 Variational Bounds

In this section, we derive variational bounds for various parameterized stochastic processes and performance metrics in the MTS model with backorders. These results will be used in subsequent sections to simplify the derivation of IPA derivatives and to establish their unbiasedness.

OSERVATION 2 *For an MTS system with the backorder rule, the stochastic differential equation (2.7) governing the outstanding order process $\{X(t)\}$ in normal operation is a special case of the SFM queue in* Wardi and Melamed (2001), *where the buffer has unlimited capacity.*

Proof: Follows from the fact that we can identify the demand arrival rate process and production rate process, respectively, with the inflow rate process and service rate process in an infinite-capacity SFM queue from Wardi and Melamed (2001). ■

For notational convenience, we define two auxiliary processes. The *extended inventory process*, $\{W(t)\}$, is defined by

$$W(t) = I(t) - B(t) = \begin{cases} I(t), & \text{if } I(t) > 0 \\ -B(t), & \text{if } I(t) = 0. \end{cases} \tag{3.1}$$

Thus, $W(t)$ determines both $I(t)$ and $B(t)$ (and vice versa). The *extended outstanding orders process*, $\{Y(t)\}$, is defined by

$$Y(t) = \begin{cases} S - I(t), & \text{if } \ I(t) > S \quad \text{(overage operation)} \\ X(t), & \text{if } \ I(t) \le S \quad \text{(normal operation).} \end{cases} \tag{3.2}$$

Observe that $Y(t)$ is negative during overage operation and non-negative during normal operation, and its time derivative satisfies

$$\frac{dY(t)}{dt^+} = \begin{cases} \alpha(t), & \text{if } \ Y(t) < 0 \\ 0, & \text{if } \ Y(t) = 0 \ \text{ and } \ \alpha(t) \le \mu(t) \\ \alpha(t) - \mu(t), & \text{otherwise.} \end{cases} \tag{3.3}$$

Furthermore, equation (2.4) implies the conservation relation

$$W(t) + Y(t) = S, \quad t \ge 0, \tag{3.4}$$

valid for each operational mode (overage and normal). The variational bounds will be shown to hold with respect to control parameters of interest at each time point, starting from an arbitrary $W(0)$.

In the remainder of the paper the tilde symbol will always indicate a realization of a random variable or a stochastic process. Accordingly, let $\{\tilde{y}(t,\theta,y)\}$ denote a sample path realization of $\{Y(t,\theta)\}$ that starts at initial condition y, so $\tilde{y}(0,\theta,y) = y$. Furthermore, when we write $\{\tilde{y}(t,\theta_1,y_1)\}$ and $\{\tilde{y}(t,\theta_2,y_2)\}$, we mean two sample path realizations for which the following conditions hold.

1. $\{\tilde{y}(t,\theta_1,y_1)\}$ and $\{\tilde{y}(t,\theta_2,y_2)\}$ are driven by the same realization, $\{\tilde{\alpha}(t)\}$, of the process $\{\alpha(t)\}$.
2. If the process $\{\mu(t)\}$ does not depend on θ, then $\{\tilde{y}(t,\theta_1,y_1)\}$ and $\{\tilde{y}(t,\theta_2,y_2)\}$ are driven by the same realization, $\{\tilde{\mu}(t)\}$, of the process $\{\mu(t)\}$. Otherwise, $\{\tilde{y}(t,\theta_1,y_1)\}$ and $\{\tilde{y}(t,\theta_2,y_2)\}$ are driven by corresponding realizations, $\{\tilde{\mu}(t,\theta_1)\}$ and $\{\tilde{\mu}(t,\theta_2)\}$ of the process $\{\mu(t)\}$, related by $\tilde{\mu}(t,\theta_1) - \tilde{\mu}(t,\theta_2) = \theta_1 - \theta_2$, in accordance with equation (2.13).

Intuitively, the two realizations have different IPA parameters and start from different initial states, $Y(0)$, but are otherwise driven by the same "randomness" in arrivals and production. In a similar vein, let $\{\tilde{\imath}(t,\theta,y)\}$ and $\{\tilde{b}(t,\theta,y)\}$ denote the realizations of the processes $\{I(t,\theta)\}$ and $\{B(t,\theta)\}$, respectively, both associated with $\{\tilde{y}(t,\theta,y)\}$.

For each realization $\{\tilde{y}(t,\theta,y)\}$ of an MTS with the backorder rule, equation (3.3) induces a partition of the interval $[0,T]$,

$$\mathcal{R}(\theta,y)=\{\mathcal{R}_1(\theta,y),\mathcal{R}_2(\theta,y),\mathcal{R}_3(\theta,y)\}, \tag{3.5}$$

where each region, $\mathcal{R}_k(\theta,y)$, $1\le k\le 3$, is defined by the condition in the corresponding k-th line of equation (3.3), namely,

$$\begin{aligned}\mathcal{R}_1(\theta,y) &= \{t\in[0,T]:\ \tilde{y}(t,\theta,y)\le 0\},\\ \mathcal{R}_2(\theta,y) &= \{t\in[0,T]:\ \tilde{y}(t,\theta,y)=0 \text{ and } \tilde{\alpha}(t)\le\tilde{\mu}(t)\},\\ \mathcal{R}_3(\theta,y) &= \{t\in[0,T]:\ [\tilde{y}(t,\theta,y)=0 \text{ and } \tilde{\alpha}(t)>\tilde{\mu}(t)] \text{ or } \tilde{y}(t,\theta,y)>0\}.\end{aligned}$$

3.1 Variational Bounds With Respect to the Base-Stock Level

In this section, the IPA parameter of interest is $S(\theta)=\theta$ for $\theta\in\Theta$, so the initial state of the extended outstanding orders process is $Y(0,\theta)=S(\theta)-W(0)$.

LEMMA 1 *For an MTS system under the backorder rule, let $y_1\le y_2$. Then for each $\theta\in\Theta$,*

$$0\le\tilde{y}(t,\theta,y_2)-\tilde{y}(t,\theta,y_1)\le y_2-y_1,\quad t\in[0,T]. \tag{3.6}$$

Proof: We show that for each θ, the difference realization, $\{\tilde{y}(t,\theta,y_2)-\tilde{y}(t,\theta,y_1)\}$, is non-increasing in t from the initial value of $y_2-y_1\ge 0$, without changing sign.

We first prove the lefthand inequality of (3.6). By assumption,

$$\tilde{y}(0,\theta,y_2)-\tilde{y}(0,\theta,y_1)=y_2-y_1\ge 0. \tag{3.7}$$

Observe that if $\tilde{y}(t^*,\theta,y_2)=\tilde{y}(t^*,\theta,y_1)$ for some time point t^*, then $\tilde{y}(t,\theta,y_2)=\tilde{y}(t,\theta,y_1)$ for all $t\ge t^*$. To see that, note that equation (3.3) implies that once the realizations synchronize, they remain synchronized thereafter. Finally, since the difference realization is continuous, it cannot change signs.

We next prove the righthand inequality of (3.6) by noting that in view of equation (3.7), it suffices to show that the derivative of the difference realization satisfies $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2)-\tilde{y}(t,\theta,y_1)]\le 0$ for all $t\ge 0$. To this end, we examine the behavior of $\frac{d}{dt^+}\tilde{y}(t,\theta,y_1)$ and $\frac{d}{dt^+}\tilde{y}(t,\theta,y_2)$ in the three regions of the partition (3.5). Informally, the proof computes $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2)-\tilde{y}(t,\theta,y_1)]$ for all pairs of regions in the partitions associated with each initial state, such that $\tilde{y}(t,\theta,y_1)$ is in one region and $\tilde{y}(t,\theta,y_2)$ is in the other. More formally, the computation covers t in all intersections of the form $\mathcal{R}_i(\theta,y_1)\bigcap\mathcal{R}_j(\theta,y_2)$, $1\le i,j\le 3$. However, the following two observations reduce substantially the number of region-pair cases to be checked. First, there is no need to check for pairs of regions with the same subscript, $i=j$, since in their intersection $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2)-\tilde{y}(t,\theta,y_1)]=0$, trivially. Second, in view of the lefthand side of (3.6), it suffices to consider only region pairs in which $\tilde{y}(t,\theta,y_1)<\tilde{y}(t,\theta,y_2)$, since obviously, $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2)-\tilde{y}(t,\theta,y_1)]=0$ when $\tilde{y}(t,\theta,y_2)=\tilde{y}(t,\theta,y_1)$.

Consequently, we need only check the following list of cases.

Case 1: $t \in \mathcal{R}_1(\theta, y_1) \bigcap \mathcal{R}_2(\theta, y_2)$. In this case, $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2) - \tilde{y}(t,\theta,y_1)] = -\tilde{\alpha}(t) \le 0$.
Case 2: $t \in \mathcal{R}_1(\theta, y_1) \bigcap \mathcal{R}_3(\theta, y_2)$. In this case, $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2) - \tilde{y}(t,\theta,y_1)] = -\tilde{\mu}(t) \le 0$.
Case 3: $t \in \mathcal{R}_2(\theta, y_1) \bigcap \mathcal{R}_3(\theta, y_2)$. In this case, $\frac{d}{dt^+}[\tilde{y}(t,\theta,y_2) - \tilde{y}(t,\theta,y_1)] = \tilde{\alpha}(t) - \tilde{\mu}(t) \le 0$, where the inequality follows from the definition of the intersection of $\mathcal{R}_2(\theta, y_1)$ and $\mathcal{R}_3(\theta, y_2)$.

The proof is complete. ■

We next derive a variational bound for an MTS system under the backorder rule, starting from an arbitrary value, w_0, of the initial extended inventory, $W(0)$.

PROPOSITION 1 *For an MTS system with the backorder rule, let $y_1 = S(\theta_1) - w_0$ and $y_2 = S(\theta_2) - w_0$, where $\theta_1, \theta_2 \in \Theta$, and w_0 is arbitrary. Then* $\max\{|\tilde{y}(t,\theta_1,y_1) - \tilde{y}(t,\theta_2,y_2)| : t \in [0,T]\} \le |\theta_1 - \theta_2|$.

Proof: Without loss of generality, assume $S(\theta_1) = \theta_1 \le \theta_2 = S(\theta_2)$, from which it follows that $0 \le y_2 - y_1 = \theta_2 - \theta_1$. Applying Lemma 1 to $\tilde{y}(t,\theta_1,y_1)$ and $\tilde{y}(t,\theta_1,y_2)$ yields

$$0 \le \tilde{y}(t,\theta_1,y_2) - \tilde{y}(t,\theta_1,y_1) \le \theta_2 - \theta_1 \quad t \in [0,T]. \tag{3.8}$$

The proposition follows immediately from (3.8), by realizing that the realization $\{\tilde{y}(t,\theta_1,y_2)\}$ is identical to $\{\tilde{y}(t,\theta_2,y_2)\}$, since they start with the same initial state, and their dynamics are independent of θ by equation (3.3). ■

COROLLARY 1 *For an MTS system with the backorder rule, let $y_1 = S(\theta_1) - w_0$ and $y_2 = S(\theta_2) - w_0$, for any w_0 and $\theta_1, \theta_2 \in \Theta$. Then,*

$$|\tilde{\imath}(t,\theta_1,y_1) - \tilde{\imath}(t,\theta_2,y_2)| \le 2\,|\theta_1 - \theta_2|, \quad t \in [0,T] \tag{3.9}$$

$$|\tilde{b}(t,\theta_1,y_1) - \tilde{b}(t,\theta_2,y_2)| \le 2\,|\theta_1 - \theta_2|, \quad t \in [0,T]. \tag{3.10}$$

Proof: Equations (3.1) and (3.4) and Proposition 1 imply that

$$|\tilde{\imath}(t,\theta_1,y_1) - \tilde{\imath}(t,\theta_2,y_2)| = |[S(\theta_1) - \tilde{y}(t,\theta_1,y_1)]^+ - [S(\theta_2) - \tilde{y}(t,\theta_2,y_2)]^+| \le 2\,|\theta_1 - \theta_2|.$$

A similar proof applies to the backorder process. ■

3.2 Variational Bounds With Respect to the Production Rate Parameter

In this section, the IPA parameter of interest is a parameter, θ, of the production rate process, $\{\mu(t,\theta)\}$, satisfying equation (2.13).

PROPOSITION 2 *Let $\theta_1, \theta_2 \in \Theta$, and assume that $Y(0,\theta_1) = Y(0,\theta_2)$. Then,*

$$\max\{|Y(t,\theta_1) - Y(t,\theta_2)| : \ t \in [0,T]\} \leq T\,|\theta_1 - \theta_2|.$$

Proof: In view of the fact that $\{Y(t,\theta_1)\}$ and $\{Y(t,\theta_2)\}$ coincide during overage operation, it suffices to consider the case $Y(0,\theta_1) = Y(0,\theta_2) \geq 0$. The proposition follows immediately from Proposition 3.2 of Wardi and Melamed (2001), because the proof there is independent of the initial state. ■

COROLLARY 2 *For any $\theta_1, \theta_2 \in \Theta$,*

$$|I(t,\theta_1) - I(t,\theta_2)| \leq T\,|\theta_1 - \theta_2|, \quad t \in [0,T]$$

and

$$|B(t,\theta_1) - B(t,\theta_2)| \leq T\,|\theta_1 - \theta_2|, \quad t \in [0,T].$$

Proof: Follows from equations (3.1) and (3.4) and Proposition 2 by a proof similar to that of Corollary 1. ■

4 IPA Derivatives

We are now in a position to derive IPA derivatives for various parameterized stochastic processes and performance metrics in the MTS model.

Let $(Q_j(\theta), R_j(\theta)), j = 1, \ldots, J(\theta)$ be the ordered extremal subintervals of $[0,\infty)$, such that $Y(t,\theta) > 0$ for all $t \in (Q_j, R_j)$; that is, the endpoints, $Q_j(\theta)$ and $R_j(\theta)$, are obtained via *inf* and *sup* functions, respectively. By convention, if any of these endpoints does not exist, then it is set to ∞.

OBSERVATION 3

$$Q_1(\theta) < R_1(\theta) < Q_2(\theta) < R_2(\theta) < \ldots < Q_{J(\theta)}(\theta) < R_{J(\theta)}(\theta), \quad w.p.1. \tag{4.1}$$

Proof: The strict inequalities will follow in equation (4.1) if we show that the equalities $Q_j(\theta) = R_j(\theta)$ and $R_j(\theta) = Q_{j+1}(\theta)$ are impossible. The first equality is ruled out because the intervals $(Q_j(\theta), R_j(\theta))$ are extremal by definition. The second equality is ruled out by part (c) of Assumption 1. ■

4.1 IPA Derivatives with Respect to the Base-Stock Level

In this section we derive IPA derivatives for the performance metrics $L_I(T,\theta)$ and $L_B(T,\theta)$ with respect to the base-stock level, $\theta = S(\theta)$, from any initial inventory state. The approach is to first derive IPA derivatives for the inventory process, $\{I(t,\theta)\}$, and backorder process, $\{B(t,\theta)\}$, and then use the results to derive the IPA derivatives of the requisite performance metrics.

ASSUMPTION 3

(*a*) *$S(\theta) = \theta$, where $\theta \in \Theta$.*
(*b*) *The processes $\{\alpha(t)\}$ and $\{\mu(t)\}$ are independent of the parameter θ.*

The following lemma identifies the time points after which the dependence of $\{Y(t,\theta)\}$ on the parameter θ ceases.

LEMMA 2 *For every $j = 1,\ldots,J(\theta)$, the process $\{Y(t,\theta):\ t > R_j(\theta)\}$ is locally independent of θ, and consequently, $\frac{d}{d\theta}Y(t,\theta) = 0$ on the event $\{t > R_j(\theta)\}$.*

Proof: It suffices to prove the lemma for $j = 1$ only, but as the proof remains unchanged, we prove it for any j. We first show that $R_j(\theta)$ is locally differentiable with respect to θ for each $j = 1,\ldots,J(\theta)$. To this end, note that $R_j(\theta)$ is not locally differentiable only when the following two simultaneous sample path events occur at time $R_j(\theta)$: the first is a jump in either of $\{\alpha(t)\}$ or $\{\mu(t)\}$ at $R_j(\theta)$, and the second corresponds to $I(R_j(\theta),\theta) = S(\theta)$ (equivalently, $Y(R_j(\theta),\theta) = 0$). However, part (c) of Assumption 1 rules out such simultaneous sample path events.

We next prove the lemma on the event $\{R_j(\theta) < t \le Q_{j+1}(\theta)\}$. Since $R_j(\theta) < Q_{j+1}(\theta)$ by Observation 3, it follows that $Q_{j+1}(\theta)$ is a jump point of $\{\alpha(t) - \mu(t)\}$, such that $\alpha(Q_{j+1}(\theta)-) \le \mu(Q_{j+1}(\theta)-)$ while $\alpha(Q_{j+1}(\theta)) > \mu(Q_{j+1}(\theta))$. Consequently, for each $j \ge 1$, $Q_{j+1}(\theta)$ is locally independent of θ and $Y(t,\theta)$ is locally independent of θ on the event $\{R_j(\theta) < t \le Q_{j+1}(\theta)\}$ and vanishes there.

It remains to prove the lemma on the event $\{t > Q_{j+1}(\theta)\}$. But this follows from the fact that $Y(Q_{j+1}(\theta),\theta)$ is locally independent of θ as shown above, and its derivative values in equation (3.3) involve only $\alpha(t)$ and $\mu(t)$, which are independent of θ. The proof is complete. ■

In the next two lemmas we make use of the hitting time, $T_S(\theta)$, defined by

$$T_S(\theta) = \begin{cases} \min\,\{t \in [0,\infty) : I(t,\theta) = S(\theta)\}, & \text{if the minimum exists} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.2}$$

LEMMA 3 *Consider an MTS system with the backorder rule on the event $\{W(0) < S(\theta)\}$ (that is, the system starts in normal operation with backorders or partial inventory). Then, for any $t \ge 0$ and $\theta \in \Theta$,*

(a) On the event $A(\theta) = \{W(0) < S(\theta)\}\bigcap\{t < T_S(\theta)\}$,

$$\frac{d}{d\theta}I(t,\theta) = \frac{d}{d\theta}B(t,\theta) = 0.$$

(b) On the event $B(\theta) = \{W(0) < S(\theta)\}\bigcap\{t > T_S(\theta)\}\bigcap\{I(t,\theta) > 0\}$,

$$\frac{d}{d\theta}I(t,\theta) = 1, \quad \frac{d}{d\theta}B(t,\theta) = 0.$$

(c) On the event $C(\theta) = \{W(0) < S(\theta)\}\bigcap\{t > T_S(\theta)\}\bigcap\{B(t,\theta) > 0\}$,

$$\frac{d}{d\theta}I(t,\theta) = 0, \quad \frac{d}{d\theta}B(t,\theta) = -1.$$

Proof: By Observation 3,

$$0 = Q_1(\theta) < T_S(\theta) = R_1(\theta) < Q_2(\theta) \quad \text{on} \quad \{W(0) < S(\theta)\}, \tag{4.3}$$

and this holds for all cases of this lemma.

To prove part (a), note that by equations (3.1) and (3.3), $\frac{dY(t,\theta)}{dt^+} = \alpha(t) - \mu(t)$ on $A(\theta)$. It follows that

$$Y(t,\theta) = Y(0,\theta) + \int_0^t [\alpha(\tau) - \mu(\tau)]\, d\tau \quad \text{on} \quad A(\theta).$$

Substituting the above into equation (3.4) implies that

$$W(t,\theta) = S(\theta) - Y(t,\theta) = W(0) - \int_0^t [\alpha(\tau) - \mu(\tau)]\, d\tau$$

is independent of θ on $A(\theta)$. It follows from equation (3.1) that $I(t,\theta) = W(t)^+$ and $B(t,\theta) = [-W(t)]^+$ are also independent of θ on $A(\theta)$, from which part (a) follows.

To prove parts (b) and (c), we apply Lemma 2 to $T_S(\theta) = R_1(\theta)$ and conclude that

$$\frac{d}{d\theta} Y(t,\theta) = 0 \text{ on } \{W(0) < S(\theta)\} \bigcap \{t > T_S(\theta)\}. \tag{4.4}$$

Next, equation (3.4) implies that on $B(\theta)$ one has $I(t,\theta) = S(\theta) - Y(t,\theta)$ and $B(t,\theta) = 0$, while on $C(\theta)$ one has $I(t,\theta) = 0$ and $B(t,\theta) = Y(t,\theta) - S(\theta)$. Parts (b) and (c) follow by differentiating these relations with respect to θ and substituting equation (4.4). ■

Lemma 4 *Consider an MTS system with the backorder rule on the event $\{W(0) > S(\theta)\}$ (that is, the system starts in overage operation). Then, for any $t \geq 0$ and $\theta \in \Theta$,*

(a) On the event $A(\theta) = \{W(0) > S(\theta)\} \bigcap \{t < T_S(\theta)\}$,

$$\frac{d}{d\theta} I(t,\theta) = \frac{d}{d\theta} B(t,\theta) = 0.$$

(b) On either of the events $B_1(\theta) = \{W(0) > S(\theta)\} \bigcap \{T_S(\theta) < Q_1(\theta)\} \bigcap \{t > T_s(\theta)\} \bigcap \{I(t,\theta) > 0\}$ or $B_2(\theta) = \{W(0) > S(\theta)\} \bigcap \{T_S(\theta) = Q_1(\theta)\} \bigcap \{t > R_1(\theta)\} \bigcap \{I(t,\theta) > 0\}$,

$$\frac{d}{d\theta} I(t,\theta) = 1, \quad \frac{d}{d\theta} B(t,\theta) = 0.$$

(c) On either of the events $C_1(\theta) = \{W(0) > S(\theta)\} \bigcap \{T_S(\theta) < Q_1(\theta)\} \bigcap \{t > T_S(\theta)\} \bigcap \{B(t,\theta) > 0\}$ or $C_2(\theta) = \{W(0) > S(\theta)\} \bigcap \{T_S(\theta) = Q_1(\theta)\} \bigcap \{t > R_1(\theta)\} \bigcap \{B(t,\theta) > 0\}$,

$$\frac{d}{d\theta} I(t,\theta) = 0, \quad \frac{d}{d\theta} B(t,\theta) = -1.$$

(d) On the event $D(\theta) = \{W(0) > S(\theta)\} \bigcap \{T_S(\theta) = Q_1(\theta)\} \bigcap \{Q_1(\theta) < t < R_1(\theta)\} \bigcap \{I(t,\theta) > 0\}$,

$$\frac{d}{d\theta} I(t,\theta) = \frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}, \quad \frac{d}{d\theta} B(t,\theta) = 0.$$

(e) On the event $E(\theta) = \{W(0) > S(\theta)\} \bigcap \{T_S(\theta) = Q_1(\theta)\} \bigcap \{Q_1(\theta) < t < R_1(\theta)\} \bigcap \{B(t,\theta) > 0\}$,

$$\frac{d}{d\theta} I(t,\theta) = 0, \quad \frac{d}{d\theta} B(t,\theta) = -\frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}.$$

Proof: To prove part (a), note that by equations (2.2) and (3.1), $W(t,\theta) = I(t,\theta)$ on $A(\theta)$. It follows from equation (2.1) that $\frac{d}{dt^+}W(t,\theta) = -\alpha(t)$ on $A(\theta)$. Therefore,

$$W(t,\theta) = W(0) - \int_0^t \alpha(\tau)\, d\tau$$

is independent of θ on $A(\theta)$. From equation (3.1) it follows that $I(t,\theta) = W(t)^+$ and $B(t,\theta) = [-W(t)]^+$ are also independent of θ on $A(\theta)$, from which part (a) follows.

To prove part (b) and (c) on the event $B_1(\theta)\bigcup C_1(\theta)$, note that by Observation 3,

$$T_S(\theta) < Q_1(\theta) < R_1(\theta) \qquad \text{on } \; B_1(\theta)\bigcup C_1(\theta). \tag{4.5}$$

Clearly, $T_S(\theta)$ is locally differentiable with respect to θ. By a proof similar to that of Lemma 2, we conclude that $Y(t,\theta)$ is locally independent of θ on the event $\{W(0) > S(\theta)\}\bigcap\{T_S(\theta) < t\}$. Consequently, parts (b) and (c) on each of the events $B_1(\theta)$ and $C_1(\theta)$ follow from equation (4.5) by the proof of parts (b) and (c) in Lemma 3.

To prove part (b) and (c) on the event $B_2(\theta)\bigcup C_2(\theta)$, note that by Observation 3,

$$T_S(\theta) = Q_1(\theta) < R_1(\theta) < Q_2(\theta) \qquad \text{on } \; B_2(\theta)\bigcup C_2(\theta). \tag{4.6}$$

Applying Lemma 2 to $R_1(\theta)$ yields that $Y(t,\theta)$ is locally independent of θ on the event $\{W(0) > S(\theta)\}\bigcap\{R_1(\theta) < t\}$. Parts (b) and (c) on each of the events $B_2(\theta)$ and $C_2(\theta)$ follow from equation (4.6) by the proof of parts (b) and (c) in Lemma 3.

To prove (d) and (e), note first that

$$\int_0^{Q_1(\theta)} \alpha(\tau)\, d\tau = W(0) - S(\theta) = W(0) - \theta,$$
$$\text{on } \; \{W(0) > S(\theta)\}\bigcap\{T_S(\theta) = Q_1(\theta)\}.$$

Differentiating the above with respect to θ yields after some manipulation,

$$\frac{d}{d\theta}Q_1(\theta) = \frac{-1}{\alpha(Q_1(\theta))}, \qquad \text{on } \; \{W(0) > S(\theta)\}\bigcap\{T_S(\theta) = Q_1(\theta)\}. \tag{4.7}$$

We next show that both $Q_1(\theta)$ and $R_1(\theta)$ depend on θ on $\{W(0) > S(\theta)\}\bigcap\{T_S(\theta) = Q_1(\theta)\}$. To this end, we write $Y(t,\theta) = \int_{Q_1(\theta)}^t [\alpha(\tau) - \mu(\tau)]\, d\tau$ on the event $D(\theta)\bigcup E(\theta)$, and then differentiate it with respect to θ. It follows that

$$\begin{aligned} &\frac{d}{d\theta}Y(t,\theta) = -[\alpha(Q_1(\theta)) - \mu(Q_1(\theta))]\,\frac{d}{d\theta}Q_1(\theta) = \frac{\alpha(Q_1(\theta)) - \mu(Q_1(\theta))}{\alpha(Q_1(\theta))}, \\ &\text{on } \; D(\theta)\bigcup E(\theta), \end{aligned} \tag{4.8}$$

where the second equality is obtained by substituting equation (4.7), and noting the inequalities $\alpha(Q_1(\theta)) > \mu(Q_1(\theta)) \geq 0$ on the event $\{W(0) > S(\theta)\}\bigcap\{Q_1(\theta) = T_S(\theta)\}$. Finally, equation (3.4) implies that on $D(\theta)$ one has $I(t,\theta) = S(\theta) - Y(t,\theta)$ and $B(t,\theta) = 0$, while on $E(\theta)$ one has $I(t,\theta) = 0$ and $B(t,\theta) = Y(t,\theta) - S(\theta)$. Parts (d) and (e) now follow by differentiating these relations with respect to θ and substituting into equation (4.8.) ■

On the event $\{S(\theta) = W(0)\}$, the situation is more complex, because the left and right derivatives of $I(t,\theta)$ and $B(t,\theta)$ with respect to θ do not coincide and must be computed separately.

We first derive the right-derivatives, $\frac{d}{d\theta^+}I(t,\theta)$ and $\frac{d}{d\theta^+}B(t,\theta)$, by borrowing from Lemma 3 and making use of the hitting time, $T_\mu(\theta)$, given by

$$T_\mu(\theta) = \begin{cases} \min\{t \in [0, Q_1(\theta)) \colon \mu(t) > \alpha(t)\}, \text{if the minimum exists on the event } \{Q_1(\theta) > 0\} \\ R_1(\theta), \qquad \text{if } \; R_1(\theta) \text{ exists on the event} \\ \qquad\qquad \{Q_1(\theta) = 0\} \bigcup [\{Q_1(\theta) > 0\} \bigcap \{\alpha(t) = \mu(t), \; t \in [0, Q_1(\theta))\}] \\ \infty, \qquad \text{otherwise.} \end{cases} \tag{4.9}$$

Lemma 5 *Consider an MTS system with the backorder rule on the event* $\{W(0) = S(\theta)\}$ *(that is, the system starts in normal operation with full inventory). Then, for any* $t \geq 0$ *and* $\theta \in \Theta$*,*

(a) On the event $A(\theta) = \{W(0) = S(\theta)\} \bigcap \{t < T_\mu(\theta)\}$*,*

$$\frac{d}{d\theta^+}I(t,\theta) = \frac{d}{d\theta^+}B(t,\theta) = 0.$$

(b) On the event $B(\theta) = \{W(0) = S(\theta)\} \bigcap \{t > T_\mu(\theta)\} \bigcap \{I(t,\theta) > 0\}$*,*

$$\frac{d}{d\theta^+}I(t,\theta) = 1, \quad \frac{d}{d\theta^+}B(t,\theta) = 0.$$

(c) On the event $C(\theta) = \{W(0) = S(\theta)\} \bigcap \{t > T_\mu(\theta)\} \bigcap \{B(t,\theta) > 0\}$*,*

$$\frac{d}{d\theta^+}I(t,\theta) = 0, \quad \frac{d}{d\theta^+}B(t,\theta) = -1.$$

Proof: Consider a perturbed system with $S(\theta + \Delta\theta) = \theta + \Delta\theta$, where $\Delta\theta > 0$. Since $W(0) = S(\theta) < S(\theta + \Delta\theta)$, it follows that the perturbed system starts in normal operation. Denote $\Delta S = S(\theta + \Delta\theta) - S(\theta)$. By Observation 3,

$$0 = Q_1(\theta + \Delta\theta) \leq T_\mu(\theta) < R_1(\theta + \Delta\theta) < Q_2(\theta + \Delta\theta) \quad \text{on } \; \{W(0) = S(\theta)\}, \tag{4.10}$$

and this holds for all cases of this lemma.

To prove part (a), note first that the event $\{T_\mu(\theta) = 0\}$ can be precluded, since it implies $A(\theta) = \emptyset$, where $\emptyset$ denotes the empty set. Otherwise, by the definition of $T_\mu(\theta)$ and equations (3.1) and (3.3), $\frac{dW(t,\theta)}{dt^+} = \mu(t) - \alpha(t)$ on $A(\theta)$, so that

$$W(t,\theta) = W(0) - \int_0^t [\alpha(\tau) - \mu(\tau)]\, d\tau \quad \text{on } A(\theta).$$

Furthermore, the definition of $T_\mu(\theta)$ ensures that both $\{W(t,\theta)\}$ and $\{W(t,\theta + \Delta\theta)\}$ are bounded from above by $S(\theta)$ on the event $A(\theta)$, so that

$$W(t,\theta + \Delta\theta) = W(0) - \int_0^t [\alpha(\tau) - \mu(\tau)]\, d\tau \quad \text{on } A(\theta).$$

We conclude that $W(t,\theta + \Delta\theta) = W(t,\theta)$ are independent of θ on $A(\theta)$. The rest of the proof of part (a) is identical to that of part (a) in Lemma 3.

To prove parts (b) and (c), observe that part (b) of Assumption 1 implies that there exists $\epsilon > 0$, such that for any $\Delta\theta \le \epsilon$,

$$R_1(\theta + \Delta\theta) = T_\mu(\theta) + \frac{\Delta S}{\mu(T_\mu(\theta)) - \alpha(T_\mu(\theta))} \qquad \text{on } \{W(0) = S(\theta)\}, \tag{4.11}$$

where the inequality $\mu(T_\mu(\theta)) - \alpha(T_\mu(\theta)) > 0$ follows from the definition of $T_\mu(\theta)$. Clearly, $T_\mu(\theta)$ is locally differentiable with respect to θ. In view of equation (4.10), it follows by a proof similar to that of Lemma 2 that $Y(t, \theta + \Delta\theta) = Y(t, \theta)$ on the event $\{W(0) = S(\theta)\} \bigcap \{t > R_1(\theta + \Delta\theta)\}$. But because $R_1(\theta + \Delta\theta) \to T_\mu(\theta)$ on $\{W(0) = S(\theta)\}$ as $\Delta\theta \to 0$ by equation (4.11), we conclude that $\frac{d}{d\theta^+} Y(t, \theta) = 0$ on the event $B(\theta) \bigcup C(\theta)$. The rest of the proof is similar to that of parts (b) and (c) in Lemma 3. ∎

We next derive the left-derivatives, $\frac{d}{d\theta^-} I(t, \theta)$ and $\frac{d}{d\theta^-} B(t, \theta)$, by borrowing from Lemma 4, and making use of the hitting time, T_α, given by

$$T_\alpha = \begin{cases} \min\{t \in [0, T] : \ \alpha(t) > 0\}, & \text{if the minimum exists} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.12}$$

Note that T_α is independent of θ.

Lemma 6 *Consider an MTS system with the backorder rule on the event $\{W(0) = S(\theta)\}$ (that is, the system starts in normal operation with full inventory). Then, for any $t \ge 0$ and $\theta \in \Theta$,*

(a) *On the event $A(\theta) = \{W(0) = S(\theta)\} \bigcap \{t < T_\alpha\}$,*

$$\frac{d}{d\theta^-} I(t, \theta) = \frac{d}{d\theta^-} B(t, \theta) = 0.$$

(b) *On either of the events $B_1(\theta) = \{W(0) = S(\theta)\} \bigcap \{T_\alpha < Q_1(\theta)\} \bigcap \{t > T_\alpha\} \bigcap \{I(t, \theta) > 0\}$* or *$B_2(\theta) = \{W(0) = S(\theta)\} \bigcap \{T_\alpha = Q_1(\theta)\} \bigcap \{t > R_1(\theta)\} \bigcap \{I(t, \theta) > 0\}$,*

$$\frac{d}{d\theta^-} I(t, \theta) = 1, \quad \frac{d}{d\theta^-} B(t, \theta) = 0.$$

(c) *On either of the events $C_1(\theta) = \{W(0) = S(\theta)\} \bigcap \{T_\alpha < Q_1(\theta)\} \bigcap \{t > T_\alpha\} \bigcap \{B(t, \theta) > 0\}$* or *$C_2(\theta) = \{W(0) = S(\theta)\} \bigcap \{T_\alpha = Q_1(\theta)\} \bigcap \{t > R_1(\theta)\} \bigcap \{B(t, \theta) > 0\}$,*

$$\frac{d}{d\theta^-} I(t, \theta) = 0, \quad \frac{d}{d\theta^-} B(t, \theta) = -1.$$

(d) *On the event $D(\theta) = \{W(0) = S(\theta)\} \bigcap \{T_\alpha = Q_1(\theta)\} \bigcap \{Q_1(\theta) < t < R_1(\theta)\} \bigcap \{I(t, \theta) > 0\}$,*

$$\frac{d}{d\theta^-} I(t, \theta) = \frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}, \quad \frac{d}{d\theta^-} B(t, \theta) = 0.$$

(e) *On the event $E(\theta) = \{W(0) = S(\theta)\} \bigcap \{T_\alpha = Q_1(\theta)\} \bigcap \{Q_1(\theta) < t < R_1(\theta)\} \bigcap \{B(t, \theta) > 0\}$,*

$$\frac{d}{d\theta^-} I(t, \theta) = 0, \quad \frac{d}{d\theta^-} B(t, \theta) = -\frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}.$$

Proof: Consider a perturbed system with $S(\theta-\Delta\theta)=\theta-\Delta\theta$, where $\Delta\theta>0$. Since by assumption $W(0)=S(\theta)>S(\theta-\Delta\theta)$, it follows that the perturbed system starts in overage operation. Denote $\Delta S=S(\theta)-S(\theta-\Delta\theta)$.

To prove part (a), note first that the event $\{T_\alpha=0\}$ can be precluded, since it implies $A(\theta)=\emptyset$. Otherwise, on the event $A(\theta)$, the perturbed system is in overage operation with no demand arrivals, so that $W(t,\theta-\Delta\theta)=W(0)=W(t,\theta)$ on the event $A(\theta)$. The rest of the proof follows from that of part (a) in Lemma 4.

To prove parts (b) and (c), observe that part (b) of Assumption 1 implies that there exists $\epsilon>0$, such that for any $\Delta\theta\leq\epsilon$,

$$T_S(\theta-\Delta\theta)=T_\alpha+\frac{\Delta S}{\alpha(T_\alpha)}\qquad \text{on } \{W(0)=S(\theta)\}, \tag{4.13}$$

where the inequality $\alpha(T_\alpha)>0$ follows from the definition of T_α.

To prove part (b) and (c) on the event $B_1(\theta)\bigcup C_1(\theta)$, note that by Observation 3,

$$T_\alpha<T_S(\theta-\Delta\theta)<Q_1(\theta)<R_1(\theta)\qquad \text{on } \ B_1(\theta)\bigcup C_1(\theta). \tag{4.14}$$

Clearly, T_α is locally differentiable with respect to θ. In view of equation (4.14), it follows from a similar proof to that of Lemma 2 that $Y(t,\theta-\Delta\theta)=Y(t,\theta)$ on the event $\{W(0)=S(\theta)\}\bigcap\{t>T_S(\theta-\Delta\theta)\}$. But because $T_S(\theta-\Delta\theta)\to T_\alpha$ on $\{W(0)=S(\theta)\}$ as $\Delta\theta\to 0$ by equation (4.13), we conclude that $\frac{d}{d\theta^-}Y(t,\theta)=0$ on the event $B_1(\theta)\bigcup C_1(\theta)$. The rest of the proof is similar to that of parts (b) and (c) in Lemma 4.

We next prove the remaining cases, namely, parts (b) and (c) for the events $B_2(\theta)$ and $C_2(\theta)$, as well as parts (d) and (e). In all these cases, by Observation 3,

$$\begin{aligned}&Q_1(\theta)=T_\alpha<T_S(\theta-\Delta\theta)=Q_1(\theta-\Delta\theta)<R_1(\theta)\\&\quad\text{on } \ B_2(\theta)\bigcup C_2(\theta)\bigcup D(\theta)\bigcup E(\theta).\end{aligned} \tag{4.15}$$

In other words, the process $\{I(t,\theta)\}$ stays at $S(\theta)$ until time T_α, at which point the arrival rate jumps, such that $\alpha(T_\alpha)>\mu(T_\alpha)$. It follows that

$$\begin{aligned}&W(T_S(\theta-\Delta\theta),\theta)=S(\theta)-\frac{\Delta S}{\alpha(T_\alpha)}\left[\alpha(T_\alpha)-\mu(T_\alpha)\right]\\&\quad\text{on } \ B_2(\theta)\bigcup C_2(\theta)\bigcup D(\theta)\bigcup E(\theta).\end{aligned} \tag{4.16}$$

But over the interval $[T_S(\theta-\Delta\theta),R_1(\theta-\Delta\theta)]$, both the original system and the perturbed system operate in their respective normal mode and are driven by identical dynamics. Therefore, the difference process $\{W(t,\theta)-W(t,\theta-\Delta\theta)\}$ is constant and positive over that interval. Consequently, by part (c) of Assumption 1, we can choose sufficiently small $\Delta\theta>0$, such that

$$\begin{aligned}&R_1(\theta-\Delta\theta)=R_1(\theta)-\frac{\Delta S\left[\alpha(T_\alpha)-\mu(T_\alpha)\right]}{\alpha(T_\alpha)\left[\mu(R_1(\theta))-\alpha(R_1(\theta))\right]}\\&\quad\text{on } \ B_2(\theta)\bigcup C_2(\theta)\bigcup D(\theta)\bigcup E(\theta).\end{aligned} \tag{4.17}$$

Furthermore, by the definition of $T_S(\theta-\Delta\theta)$ and equation (4.16), $W(t,\theta)-W(t,\theta-\Delta\theta)=\frac{\Delta S\,\mu(T_\alpha)}{\alpha(T_\alpha)}$, $t\in[T_S(\theta-\Delta\theta),R_1(\theta-\Delta\theta)]$. Combining this with equation (3.4), we conclude that

$$Y(t,\theta)-Y(t,\theta-\Delta\theta)=\Delta S\frac{\alpha(T_\alpha)-\mu(T_\alpha)}{\alpha(T_\alpha)} \qquad \text{on } \{T_S(\theta-\Delta\theta)\le t\le R_1(\theta-\Delta\theta)\}. \tag{4.18}$$

Moreover,

$$Y(t,\theta-\Delta\theta)=Y(t,\theta) \qquad \text{on } \{t>R_1(\theta)\}, \tag{4.19}$$

due to part (c) of Assumption 1. Next, send $\Delta S=\Delta\theta\to 0$ in equations (4.13) and (4.17), yielding $T_S(\theta-\Delta\theta)\to T_\alpha$ and $R_1(\theta-\Delta\theta)\to R_1(\theta)$, respectively. The left derivative $\frac{d}{d\theta^-}Y(t,\theta)$ can now be readily computed on events $B_2(\theta)$ and $C_2(\theta)$ from equation (4.19), and on events $D(\theta)$ and $E(\theta)$ from equation (4.18). Finally, the rest of the proof of parts (b) and (c) follows similarly to the proof of parts (b) and (c) of Lemma 4, while that of parts (d) and (e) follows similarly to the proof of parts (d) and (e) of Lemma 4. ■

THEOREM 1 *W.p.1, the IPA derivatives of the inventory time average with respect to the base-stock level are given for all $T>0$ and $\theta\in\Theta$ as follows:*

(a) *On the event* $\{W(0)<S(\theta)\}$,

$$\frac{d}{d\theta}L_I(T,\theta)=1_{\{T_S(\theta)<T\}}\,\frac{1}{T}\int_{T_S(\theta)}^{T}1_{\{I(t,\theta)>0\}}\,dt. \tag{4.20}$$

(b) *On the event* $\{W(0)>S(\theta)\}\bigcap\{T_S(\theta)<Q_1(\theta)\}$,

$$\frac{d}{d\theta}L_I(T,\theta)=1_{\{T_S(\theta)<T\}}\,\frac{1}{T}\int_{T_S(\theta)}^{T}1_{\{I(t,\theta)>0\}}\,dt. \tag{4.21}$$

(c) On the event $\{W(0)>S(\theta)\}\bigcap\{T_S(\theta)=Q_1(\theta)\}$,

$$\frac{d}{d\theta}L_I(T,\theta)=1_{\{Q_1(\theta)<T\}}\,\frac{1}{T}\left[\frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}\int_{Q_1(\theta)}^{\min\{R_1(\theta),T\}}1_{\{I(t,\theta)>0\}}\,dt + 1_{\{R_1(\theta)<T\}}\int_{R_1(\theta)}^{T}1_{\{I(t,\theta)>0\}}\,dt\right]. \tag{4.22}$$

(d) *On the event* $\{W(0)=S(\theta)\}$,

$$\frac{d}{d\theta^+}L_I(T,\theta)=1_{\{T_\mu(\theta)<T\}}\,\frac{1}{T}\int_{T_\mu(\theta)}^{T}1_{\{I(t,\theta)>0\}}\,dt. \tag{4.23}$$

(e) *On the event* $\{W(0)=S(\theta)\}\bigcap\{T_\alpha<Q_1(\theta)\}$,

$$\frac{d}{d\theta^-}L_I(T,\theta)=1_{\{T_\alpha<T\}}\,\frac{1}{T}\int_{T_\alpha}^{T}1_{\{I(t,\theta)>0\}}\,dt. \tag{4.24}$$

(f) *On the event* $\{W(0) = S(\theta)\} \bigcap \{T_\alpha = Q_1(\theta)\}$,

$$\frac{d}{d\theta^-} L_I(T,\theta) = 1_{\{Q_1(\theta)<T\}} \tag{4.25}$$

$$\frac{1}{T}\left[\frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}\int_{Q_1(\theta)}^{\min\{R_1(\theta),T\}} 1_{\{I(t,\theta)>0\}}\,dt + 1_{\{R_1(\theta)<T\}}\int_{R_{1(\theta)}}^{T} 1_{\{I(t,\theta)>0\}}dt\right].$$

Proof: We show that Leibniz's rule can be applied to equation (2.10) yielding

$$\frac{d}{d\theta^\pm} L_I(T,\theta) = \frac{1}{T}\frac{d}{d\theta^\pm}\int_0^T I(t,\theta)\,dt = \frac{1}{T}\int_0^T \frac{d}{d\theta^\pm} I(t,\theta)\,dt. \tag{4.26}$$

To this end, note that Assumption 1 and Lemmas 3–6 ensure that w.p.1., the sided derivatives $\frac{dI(t,\theta)}{d\theta^\pm}$ exist and are finite over the interval $[0,T]$, except possibly for a finite number of time points. Furthermore, since the starting time and ending time in the integral of equation (4.26) are independent of θ, it follows from Corollary 1 that the differentiation and the integration operations commute there. The theorem now follows by substituting the values of the derivatives computed in Lemmas 3–6 into equation (4.26). ■

THEOREM 2 *W.p.1, the IPA derivatives of the backorder time average with respect to the base-stock level are given for all* $T > 0$ *and* $\theta \in \Theta$ *as follows:*

(a) *On the event* $\{W(0) < S(\theta)\}$,

$$\frac{d}{d\theta} L_B(T,\theta) = -1_{\{T_S(\theta)<T\}}\,\frac{1}{T}\int_{T_S(\theta)}^{T} 1_{\{B(t,\theta)>0\}}\,dt. \tag{4.27}$$

(b) *On the event* $\{W(0) > S(\theta)\} \bigcap \{T_S(\theta) < Q_1(\theta)\}$,

$$\frac{d}{d\theta} L_B(T,\theta) = -1_{\{T_S(\theta)<T\}}\,\frac{1}{T}\int_{T_S(\theta)}^{T} 1_{\{B(t,\theta)>0\}}\,dt. \tag{4.28}$$

(c) *On the event* $\{W(0) > S(\theta)\} \bigcap \{T_S(\theta) = Q_1(\theta)\}$,

$$\frac{d}{d\theta} L_B(T,\theta) = -1_{\{Q_1(\theta)<T\}}\,\frac{1}{T}\left[\frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))}\int_{Q_1(\theta)}^{\min\{R_1(\theta),T\}} 1_{\{B(t,\theta)>0\}}\,dt \right. \tag{4.29}$$

$$\left. + 1_{\{R_1(\theta)<T\}}\int_{R_1(\theta)}^{T} 1_{\{B(t,\theta)>0\}}\,dt\right].$$

(d) *On the event* $\{W(0) = S(\theta)\}$,

$$\frac{d}{d\theta^+} L_B(T,\theta) = -1_{\{T_\mu(\theta)<T\}}\,\frac{1}{T}\int_{T_\mu(\theta)}^{T} 1_{\{B(t,\theta)>0\}}\,dt. \tag{4.30}$$

(e) *On the event* $\{W(0) = S(\theta)\} \bigcap \{T_\alpha < Q_1(\theta)\}$,

$$\frac{d}{d\theta^-} L_B(T,\theta) = -1_{\{T_\alpha<T\}}\,\frac{1}{T}\int_{T_\alpha}^{T} 1_{\{B(t,\theta)>0\}}\,dt. \tag{4.31}$$

(f) *On the event* $\{W(0) = S(\theta)\} \bigcap \{T_\alpha = Q_1(\theta)\}$,

$$\frac{d}{d\theta^-} L_B(T,\theta) = -1_{\{Q_1(\theta)<T\}} \frac{1}{T} \left[\frac{\mu(Q_1(\theta))}{\alpha(Q_1(\theta))} \int_{Q_1(\theta)}^{min\{R_1(\theta),T\}} 1_{\{B(t,\theta)>0\}}\, dt \right. \\ \left. + 1_{\{R_1(\theta)<T\}} \int_{R_1(\theta)}^{T} 1_{\{B(t,\theta)>0\}}\, dt \right]. \tag{4.32}$$

Proof: Similar to the proof of Theorem 1. ■

The transient results in Theorems 1 and 2 extend readily to long-run results as $T \to \infty$.

Corollary 3 *Let H be any of the hitting times $T_S(\theta)$, $R_1(\theta)$, $T_\mu(\theta)$or T_α. Then, the corresponding parts in Theorems 1 and 2 assume the common forms*

$$\lim_{T\to\infty} \frac{d}{d\theta^\pm} L_I(T,\theta) = 1_{\{H<\infty\}} \lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\{I(t,\theta)>0\}}\, dt \tag{4.33}$$

$$\lim_{T\to\infty} \frac{d}{d\theta^\pm} L_B(T,\theta) = -1_{\{H<\infty\}} \lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\{B(t,\theta)>0\}}\, dt \tag{4.34}$$

when the limits exist on the corresponding events. ■

Thus, for T large enough, equations (4.33) and (4.34) provide computationally handy approximations of the requisite IPA derivatives. Note further that under ergodic conditions, we can interpret the limits in equations (4.33) and (4.34) as the long run (equilibrium) probabilities, $\Pr\{I(\theta) > 0\}$ and $\Pr\{B(\theta) > 0\}$, respectively, where $I(\theta)$ and $B(\theta)$ are random variables having the requisite long-run distributions.

Finally, we show that the IPA derivatives of Theorems 1 and 2 are unbiased.

Theorem 3 *Under Assumptions 1 and 3, the sided IPA derivatives with respect to the base level parameter, $\frac{d}{d\theta^\pm} L_I(T,\theta)$ and $\frac{d}{d\theta^\pm} L_B(T,\theta)$, are unbiased for all $T > 0$ and $\theta \in \Theta$.*

Proof: Theorems 1 and 2 ensure that for all $T > 0$, Condition (a) of Fact 1 is satisfied for both $L_I(T,\theta)$ and $L_B(T,\theta)$. On any event of the form $\{W(0) = w_0\}$, one has by definition,

$$I(t,\theta) = \tilde{i}(t,\theta,y) \quad \text{and} \quad B(t,\theta) = \tilde{b}(t,\theta,y),$$

where $y = S(\theta) - w_0$. Thus, for any $\theta_1, \theta_2 \in \Theta$,

$$\begin{aligned} |L_I(T,\theta_1) - L_I(T,\theta_2)| &= \left| \frac{1}{T} \int_0^T [I(t,\theta_1) - I(t,\theta_2)]\, dt \right| \\ &\le \frac{1}{T} \int_0^T |I(t,\theta_1) - I(t,\theta_2)|\, dt \;\le\; 2|\theta_1 - \theta_2|, \end{aligned} \tag{4.35}$$

where the second inequality is a consequence of Corollary 1. Similarly,

$$|L_B(T,\theta_1) - L_B(T,\theta_2)| \;\le\; 2|\theta_1 - \theta_2|. \tag{4.36}$$

Equations (4.35) and (4.36) establish that Condition (b) of Fact 1 holds for both $L_I(T,\theta)$ and $L_B(T,\theta)$, thereby completing their proof of unbiasedness. ■

4.2 IPA Derivatives with Respect to the Production Rate Parameter

In this section we derive IPA derivatives for the performance metrics $L_I(T,\theta)$ and $L_B(T,\theta)$ with respect to a production rate parameter θ. The approach is to first derive IPA derivatives for the inventory process, $\{I(t,\theta)\}$, and backorder process, $\{B(t,\theta)\}$, and then use the results to derive the IPA derivatives of the requisite performance metrics.

ASSUMPTION 4

(*a*) *The production rate process* $\{\mu(t,\theta)\}$ *is subject to equation* (2.13).
(*b*) *The process* $\{\alpha(t)\}$*and the base-stock level, S, are independent of* θ.

We point out that unlike Wardi et al. (2002) and Zhao and Melamed (2004), Assumption 4 admits the possibility that the sided IPA derivatives do not coincide. Indeed, this could happen on events of the form $\{I(t,\theta)=S\}\bigcap\{\alpha(t)=\mu(t,\theta)\}$. The probabilities of these events are generally not zero because $I(t,\theta)=S$ could hold true for an extended period of time, and by part (a) of Assumption 1, $\{\alpha(t)\}$ and $\{\mu(t,\theta)\}$ have sample paths that are piecewise-constant w.p.1.

In this section, we may assume without loss of generality that $W(0)\le S$, since the replenishment process, $\{\rho(t)\}$, vanishes during overage operation, so that the value of θ has no effect on the state of the system until it enters normal mode. Consequently, we may assume $Y(t,\theta)\ge 0$ for all $t\in[0,T]$ without loss of generality.

For notational convenience, define $R_0(\theta)=0$, so that we can write $Y(t,\theta)=0$ on the extremal intervals $[R_j(\theta),Q_{j+1}(\theta)]$, $j=0,1,\cdots,J(\theta)-1$. For each positive-length interval $[R_j(\theta),Q_{j+1}(\theta)]$, let $[E_{j,k}(\theta),F_{j,k}(\theta)]$, $k=1,2,\cdots,K_j(\theta)$ be their extremal subintervals on which $\alpha(t)=\mu(t,\theta)$, if they exist. Otherwise, if they do not exist, we set $K_j(\theta)=0$.

LEMMA 7

(*a*) *For* $j=1,\cdots,J(\theta)$, $Q_j(\theta)$ *is locally independent of* θ *in a right neighborhood of* θ.
(*b*) *For all* $j=0,1,\cdots,J(\theta)-1$, *if* $K_j(\theta)=0$ *or if* $K_j(\theta)>0$ *and* $F_{j,K_j(\theta)}(\theta)<Q_{j+1}(\theta)$, *then the representation*

$$Y(t,\theta-\Delta\theta)=\int_{Q_{j+1}(\theta)}^{t}[\alpha(\tau)-\mu(\tau,\theta-\Delta\theta)]\,d\tau,\quad t\in(Q_{j+1}(\theta),R_{j+1}(\theta)),\quad(4.37)$$

holds for any $\theta\in\Theta$ *and sufficiently small* $\Delta\theta>0$.
(*c*) *For all* $j=0,1,\cdots,J(\theta)-1$, $k=1,2,\cdots,K_j(\theta)$, *the representation*

$$Y(t,\theta-\Delta\theta)=\int_{E_{j,k}(\theta)}^{t}[\alpha(\tau)-\mu(\tau,\theta-\Delta\theta)]\,d\tau,\quad t\in(E_{j,k}(\theta),F_{j,k}(\theta)),\quad(4.38)$$

holds for any $\theta\in\Theta$ *and sufficiently small* $\Delta\theta>0$.

Proof: To prove part (a), we first prove it for $j = 1$ by considering the events $\{I(0) < S\}$ and $\{I(0) = S\}$ separately. On the event $\{I(0) < S\}$, $Q_1(\theta) = 0$, which is independent of θ. On the event $\{I(0) = S\}$, note that $\alpha(t) - \mu(t, \theta + \Delta\theta) < 0$ for $\Delta\theta > 0$ in the interval $[0, Q_1(\theta))$, whence $Y(t, \theta + \Delta\theta) = 0$ in the same interval. Furthermore, $Q_1(\theta)$ corresponds to a jump in $\{\alpha(t) - \mu(t, \theta)\}$ from non-positive value to a positive value. By part (b) of Assumption 1, such jumps are independent of θ, whence $Q_1(\theta)$ is locally independent of θ in a right neighborhood of θ.

We next consider any $j > 1$, and note that for $\Delta\theta > 0$, $\alpha(t) - \mu(t, \theta + \Delta\theta) < 0$ in the interval $[R_{j-1}, Q_j(\theta))$. Hence, $Y(t, \theta + \Delta\theta) = 0$ in the same interval, and combining this fact with part (c) of Assumption 1, we conclude that $Q_j(\theta)$ is locally differentiable with respect to θ in a right neighborhood of θ. The rest of the proof for $j > 1$ is the same as that for $j = 1$ above.

To prove part (b), we first prove it for $j = 0$ by considering the events $\{I(0) < S\}$ and $\{I(0) = S\}$ separately. On the event $\{I(0) < S\}$, $Q_1(\theta) = 0$, and it follows from equation (3.3) that

$$Y(t, \theta - \Delta\theta) = \int_{Q_1(\theta)}^{t} [\alpha(\tau) - \mu(\tau, \theta - \Delta\theta)]\, d\tau, \quad t \in (Q_1(\theta), R_1(\theta)), \tag{4.39}$$

for any $\theta \in \Theta$ and sufficiently small $\Delta\theta > 0$. On the event $\{I(0) = S\}$, when $K_0(\theta) = 0$ or when $K_0(\theta) > 0$ and $F_{0,K_0(\theta)}(\theta) < Q_1(\theta)$, then $Q_1(\theta)$ corresponds to a jump in $\{\alpha(t) - \mu(t, \theta)\}$ from a negative value to a positive one. By part (b) of Assumption 1, such jumps are independent of θ, so that equation (4.39) again holds.

We next consider any $j > 0$, and note that when $K_j(\theta) = 0$ or when $K_j(\theta) > 0$ and $F_{j,K_j(\theta)}(\theta) < Q_{j+1}(\theta)$, then $Q_{j+1}(\theta)$ corresponds to a jump in $\{\alpha(t) - \mu(t, \theta)\}$ from a negative value to a positive one. The rest of the proof for $j > 0$ is the same as that for $j = 0$ above.

To prove part (c), note that each $E_{j,k}(\theta)$ corresponds to a jump in $\{\alpha(t) - \mu(t, \theta)\}$ from a negative value to zero. By part (c) of Assumption 1, for sufficiently small $\Delta\theta > 0$, each $E_{j,k}(\theta)$ is the left endpoint of an extremal interval in which $Y(t, \theta - \Delta\theta) > 0$, and that left endpoint is locally independent of θ in a left neighborhood of θ. Therefore, $Y(E_{j,k}(\theta), \theta - \Delta\theta) = 0$ and the representation in equation (4.38) follows from equation (3.3). ■

For each interval $[R_j(\theta), Q_{j+1}(\theta)]$, $j = 0, 1, \cdots, J(\theta) - 1$, define

$$P^*_{j+1}(\theta) = \begin{cases} Q_{j+1}(\theta), \textit{if } K_j(\theta) = 0 \text{ or } K_j(\theta) > 0 \text{ and } F_{j,K_j(\theta)}(\theta) < Q_{j+1}(\theta) \\ E_{j,K_j(\theta)}, \text{ otherwise.} \end{cases} \tag{4.40}$$

Part (c) of Assumption 1 ensures that the $P^*_{j+1}(\theta)$ exists.

We first derive the right-derivative, $\frac{d}{d\theta^+} Y(t, \theta)$.

Lemma 8 *Consider an MTS system with the backorder rule. Then, for any $t \geq 0$ and $\theta \in \Theta$,*

(*a*) *On the events $A_j(\theta) = \{R_j(\theta) < t < Q_{j+1}(\theta)\}$, $j = 0, 1, \cdots, J(\theta) - 1$,*

$$\frac{d}{d\theta^+} Y(t, \theta) = 0.$$

(*b*) *On the events* $B_j(\theta) = \{Q_j(\theta) < t < R_j(\theta)\}$, $j = 1, \cdots, J(\theta)$,

$$\frac{d}{d\theta^+} Y(t,\theta) = -\int_{Q_j(\theta)}^{t} d\tau = Q_j(\theta) - t.$$

Proof: To prove part (a), note that $\alpha(t) - \mu(t,\theta) < 0$ on the events $A_j(\theta)$ in a right neighborhood of θ. Therefore, $Y(t,\theta) = 0$ on each event $A_j(\theta)$ in a right neighborhood of θ. Part (a) now follows from part (c) of Assumption 1.

To prove part (b), we consider the following two cases.

Case 1: event $B_1(\theta) \bigcap \{Q_1(\theta) = 0\}$. On this event, equation (3.3) implies

$$Y(t,\theta) = Y(0) + \int_0^t [\alpha(\tau) - \mu(\tau,\theta)]\, d\tau,$$

where the initial state $Y(0) = S - W(0)$ is independent of θ. Taking right derivatives with respect to θ in the equation above yields

$$\frac{dY(t,\theta)}{d\theta^+} = -\int_0^t d\tau = -t,$$

by Assumption 4 and equation (2.13), which is the requisite result.

Case 2: event $B_1(\theta) \bigcap \{Q_1(\theta) > 0\}$ or events $B_j(\theta)$, $j > 1$. On any of these events, equation (3.3) implies

$$Y(t,\theta) = \int_{Q_j(\theta)}^{t} [\alpha(\tau) - \mu(\tau,\theta)]\, d\tau,$$

since $Y(Q_j(\theta),\theta) = 0$. Taking right derivatives with respect to θ in the equation above yields

$$\frac{dY(t,\theta)}{d\theta^+} = -[\alpha(Q_j(\theta+)) - \mu(Q_j(\theta+),\theta)]\, \frac{d}{d\theta^+} Q_j(\theta) - \int_{Q_j(\theta)}^{t} d\tau,$$

by Assumption 4 and equation (2.13). By part (a) of Lemma 7, $\frac{d}{d\theta^+} Q_j(\theta) = 0$, thereby yielding the requisite result. ∎

We next derive the left-derivative, $\frac{d}{d\theta^-} Y(t,\theta)$. For notational convenience, we shall use the conventions $F_{j,0}(\theta) = R_j(\theta)$ and $E_{j,K_j(\theta)+1}(\theta) = Q_{j+1}(\theta)$, for all $j = 0, \ldots, J(\theta) - 1$.

Lemma 9 *Consider an MTS system with the backorder rule. Then, for any* $t \geq 0$ *and* $\theta \in \Theta$,

(*a*) *On the events* $A_{j,k}(\theta) = \{F_{j,k}(\theta) < t < E_{j,k+1}(\theta)\}$, $j = 0, 1, \ldots, J(\theta) - 1$, $k = 0, 1, \ldots, K_j(\theta)$,

$$\frac{d}{d\theta^-} Y(t,\theta) = 0.$$

(*b*) *On the events* $B_{j,k}(\theta) = \{E_{j,k}(\theta) < t < F_{j,k}(\theta)\}$, $j = 0, 1, \ldots, J(\theta) - 1$, $k = 1, \ldots, K_j(\theta)$,

$$\frac{d}{d\theta^-} Y(t, \theta) = -\int_{E_{j,k}(\theta)}^{t} d\tau = E_{j,k}(\theta) - t.$$

(*c*) *On the events* $C_j(\theta) = \{Q_j(\theta) < t < R_j(\theta)\}$, $j = 1, \ldots, J(\theta)$,

$$\frac{d}{d\theta^-} Y(t, \theta) = -\int_{P_j^*(\theta)}^{t} d\tau = P_j^*(\theta) - t.$$

Proof: To prove part (a), note that $\alpha(t) - \mu(t, \theta) < 0$ on the events $A_{j,k}(\theta)$ in a left neighborhood of θ. Therefore, $Y(t, \theta) = 0$ on each event $A_{j,k}(\theta)$ in a left neighborhood of θ. Part (a) now follows from part (c) of Assumption 1.

To prove part (b), consider $\theta - \Delta\theta$ for $\Delta\theta > 0$. On any event $B_{j,k}(\theta)$, one has

$$\alpha(t) = \mu(t, \theta) \tag{4.41}$$

$$Y(t, \theta) = 0 \tag{4.42}$$

$$Y(t, \theta - \Delta\theta) = \int_{E_{j,k}(\theta)}^{t} [\alpha(\tau) - \mu(\tau, \theta - \Delta\theta)] d\tau = \int_{E_{j,k}(\theta)}^{t} [\mu(\tau, \theta) - \mu(\tau, \theta - \Delta\theta)] d\tau, \tag{4.43}$$

where equation (4.41) and (4.42) follow by definition, and in equation (4.43) the first equality follows for sufficiently small $\Delta\theta > 0$ from part (c) of Lemma 7, and the second follows from equation (4.41), equations (4.42) and (4.43) now imply

$$\lim_{\Delta\theta \to 0} \frac{Y(t, \theta) - Y(t, \theta - \Delta\theta)}{\Delta\theta} = -\int_{E_{j,k}(\theta)}^{t} d\tau,$$

which is the requisite result for part (b).

To prove part (c), consider again $\theta - \Delta\theta$ for $\Delta\theta > 0$. On the events $C_j(\theta)$, equation (3.3) implies

$$Y(t, \theta) = \int_{Q_j(\theta)}^{t} [\alpha(\tau) - \mu(\tau, \theta)] \, d\tau. \tag{4.44}$$

To complete the proof, we consider two cases.

Case 1: event $C_j(\theta) \bigcap \{P_j^*(\theta) < Q_j(\theta)\}$. On this event, $Y(P_j^*(\theta), \theta - \Delta\theta) = 0$ for sufficiently small $\Delta\theta > 0$ by equation (4.40) and part (c) of Lemma 7. It follows from equation (3.3) and part (c) of Lemma 7 that

$$\begin{aligned} Y(t, \theta - \Delta\theta) &= \int_{P_j^*(\theta)}^{t} [\alpha(\tau) - \mu(\tau, \theta - \Delta\theta)] \, d\tau \\ &= \int_{P_j^*(\theta)}^{Q_j(\theta)} [\alpha(\tau) - \mu(\tau, \theta - \Delta\theta)] \, d\tau + \int_{Q_j(\theta)}^{t} [\alpha(\tau) - \mu(\tau, \theta - \Delta\theta)] \, d\tau \\ &= \int_{P_j^*(\theta)}^{Q_j(\theta)} [\mu(\tau, \theta) - \mu(\tau, \theta - \Delta\theta)] \, d\tau + \int_{Q_j(\theta)}^{t} [\alpha(\tau) - \mu(\tau, \theta - \Delta\theta)] \, d\tau, \end{aligned} \tag{4.45}$$

where the third equality is due to fact that on the event $\{P_j^*(\theta) < t < Q_j(\theta)\}$, one has $\alpha(t) = \mu(t, \theta)$. Subtracting equation (4.45) from equation (4.44) yields

$$\begin{aligned} Y(t,\theta) - Y(t,\theta-\Delta\theta) &= -\int_{P_j^*(\theta)}^{Q_j(\theta)} [\mu(\tau,\theta) - \mu(\tau,\theta-\Delta\theta)]\, d\tau - \int_{Q_j(\theta)}^{t} [\mu(\tau,\theta) \\ &\quad - \mu(\tau,\theta-\Delta\theta)]\, d\tau \\ &= -\int_{P_j^*(\theta)}^{t} [\mu(\tau,\theta) - \mu(\tau,\theta-\Delta\theta)]\, d\tau, \end{aligned}$$

whence,

$$\lim_{\Delta\theta\to 0} \frac{Y(t,\theta) - Y(t,\theta-\Delta\theta)}{\Delta\theta} = -\int_{P_j^*(\theta)}^{t} d\tau,$$

which is the requisite result.

Case 2: event $C_j(\theta) \bigcap \{P_j^*(\theta) = Q_j(\theta)\}$. On this event, equation (4.40) and part (b) of Lemma 7 imply that for sufficiently small $\Delta\theta > 0$,

$$Y(t,\theta-\Delta\theta) = \int_{Q_j(\theta)}^{t} [\alpha(\tau) - \mu(\tau,\theta-\Delta\theta)]\, d\tau.$$

The rest of the proof is a simplified version of that in Case 1 above.

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For each $j = 1, 2, \ldots, J(\theta)$, partition the interval $(Q_j(\theta), R_j(\theta))$ by two sequences of subintervals as follows:

1. Define $(G_{j,m}(\theta), H_{j,m}(\theta))$, $m = 1, 2, \ldots, M_j(\theta)$, to be the extremal subintervals of $(Q_j(\theta), R_j(\theta))$ in which $I(t,\theta) > 0$
2. Define $(U_{j,n}(\theta), V_{j,n}(\theta))$, $n = 1, 2, \ldots, N_j(\theta)$, to be the extremal subintervals $(Q_j(\theta), R_j(\theta))$ in which $B(t,\theta) > 0$.

By Assumption 1, $M_j(\theta)$ and $N_j(\theta)$ are finite for all j, w.p.1.

We shall need the following horizon-dependent random indices. The restriction of $J(\theta)$ to a finite time horizon $[0, T]$ is

$$J(T,\theta) = \begin{cases} \max\{j \ge 1\colon Q_j(\theta) \le T\}, & \text{if it exists} \\ 0, & \text{otherwise.} \end{cases} \tag{4.46}$$

The restriction of $K_j(\theta)$ to a finite time horizon $[0, T]$ is

$$K_j(T,\theta) = \begin{cases} \max\{k \ge 1\colon E_{j,k}(\theta) \le T\}, & \text{if it exists} \\ 0, & \text{otherwise.} \end{cases} \tag{4.47}$$

The restriction of $M_j(\theta)$ to a finite time horizon $[0, T]$ is

$$M_j(T,\theta) = \begin{cases} \max\{m \ge 1\colon G_{j,m}(\theta) \le T\}, & \text{if it exists} \\ 0, & \text{otherwise.} \end{cases} \tag{4.48}$$

The restriction of $N_j(\theta)$ to a finite time horizon $[0, T]$ is

$$N_j(T,\theta) = \begin{cases} \max\{n \geq 1\colon U_{j,n}(\theta) \leq T\}, & \text{if it exists} \\ 0, & \text{otherwise.} \end{cases} \tag{4.49}$$

THEOREM 4 *W.p.1, the IPA derivatives of the inventory time average with respect to the production rate parameter are given for all $T > 0$ and $\theta \in \Theta$ as follows:*

$$\frac{d}{d\theta^+} L_I(T,\theta) = \frac{1}{2T} \sum_{j=1}^{J(T,\theta)} \sum_{m=1}^{M_j(T,\theta)} [\min\{H_{j,m}(\theta), T\} - G_{j,m}(\theta)]\,[\min\{H_{j,m}(\theta), T\} + G_{j,m}(\theta) - 2Q_j(\theta)] \tag{4.50}$$

$$\frac{d}{d\theta^-} L_I(T,\theta) = \frac{1}{2T} \sum_{j=1}^{J(T,\theta)} \sum_{m=1}^{M_j(T,\theta)} [\min\{H_{j,m}(\theta), T\} - G_{j,m}(\theta)]\,[\min\{H_{j,m}(\theta), T\} + G_{j,m}(\theta) - 2P_j^*(\theta)] + \frac{1}{2T} \sum_{j=0}^{J(T,\theta)} \sum_{k=1}^{K_j(T,\theta)} [\min\{F_{j,k}(\theta), T\} - E_{j,k}(\theta)]^2 \tag{4.51}$$

Proof: We show that Leibniz's rule can be applied to equation (2.10) yielding

$$\frac{d}{d\theta^\pm} L_I(T,\theta) = \frac{1}{T} \frac{d}{d\theta^\pm} \int_0^T I(t,\theta)\,dt = \frac{1}{T} \int_0^T \frac{d}{d\theta^\pm} I(t,\theta)\,dt. \tag{4.52}$$

To this end, note that $I(t,\theta) = 0$ on the events $\{U_{j,n} < t < V_{j,n}\}$, $j = 1,\ldots,J(\theta)$, $n = 1,\ldots,N_j(\theta)$, so that $\frac{d}{d\theta} I(t,\theta) = 0$ by part (c) of Assumption 1. From equation (3.4), $I(t,\theta) = S - Y(t,\theta)$ on all other events. Consequently, by part (b) of Assumption 4 we can write

$$\frac{d}{d\theta^\pm} I(t,\theta) = -\frac{d}{d\theta^\pm} Y(t,\theta).$$

Next, note that Assumption 1 and Lemmas 8 and 9 ensure that w.p.1., the sided derivatives $\frac{d}{d\theta^\pm} I(t,\theta)$ exist and are finite over the interval $[0, T]$, except possibly for a finite number of time points. Furthermore, since the starting time and ending time in the integral of equation (4.52) are independent of θ, it follows from Corollary 2 that the differentiation and the integration operations commute there. The theorem now follows by substituting the values of the derivatives computed in Lemmas 8 and 9 into equation (4.52). ■

THEOREM 5 *W.p.1, the IPA derivatives of the backorder time average with respect to the production rate parameter are given for all $T > 0$ and $\theta \in \Theta$ as follows:*

$$\frac{d}{d\theta^+} L_B(T,\theta) = -\frac{1}{2T} \sum_{j=1}^{J(T,\theta)} \sum_{n=1}^{N_j(T,\theta)} [\min\{V_{j,n}(\theta), T\} - U_{j,n}(\theta)][\min\{\mathrm{V}_{\mathrm{j,n}}(\theta), \mathrm{T}\} + \mathrm{U}_{\mathrm{j,n}}(\theta) - 2\mathrm{Q}_{\mathrm{j}}(\theta)] \tag{4.53}$$

$$\frac{d}{d\theta^-}L_B(T,\theta) = -\frac{1}{2T}\sum_{j=1}^{J(T,\theta)}\sum_{n=1}^{N_j(T,\theta)}[\min\{V_{j,n}(\theta),T\} - U_{j,n}(\theta)][\min\{V_{j,n}(\theta),T\} + U_{j,n}(\theta) - 2P_j^*(\theta)] \tag{4.54}$$

Proof: First note that on the events $\{U_{j,n} < t < V_{j,n}\}$ for all $j = 1,\ldots,J(T,\theta)$, $n = 1,\ldots,N_j(T,\theta)$, one has $B(t,\theta) = Y(t,\theta) - S$ by equation (3.4). Consequently, by part (b) of Assumption 4 we can write

$$\frac{d}{d\theta^\pm}B(t,\theta) = \frac{d}{d\theta^\pm}Y(t,\theta).$$

On all other events, $B(t,\theta) = 0$ trivially, so it follows from part (c) of Assumption 1 that $\frac{d}{d\theta^\pm}B(t,\theta) = 0$ there. The rest of the proof is similar to that of Theorem 4. ■

Clearly, if $\alpha(t) < \mu(t,\theta)$ for all $t \in [0,T]$ satisfying $Y(t,\theta) = 0$, then the left and right derivatives of L_I are identical, and so are those of L_B.

Finally, we show that the IPA derivatives with respect to the production rate parameter are unbiased.

THEOREM 6 *Under Assumptions 1 and 4, the sided IPA derivatives with respect to the production rate parameter,* $\frac{d}{d\theta^\pm}L_I(T,\theta)$ *and* $\frac{d}{d\theta^\pm}L_B(T,\theta)$*, are unbiased for all* $T > 0$ *and* $\theta \in \Theta$.

Proof: Theorems 4 and 5 ensure that for all $T > 0$, Condition (a) of Fact 1 is satisfied for both $L_I(T,\theta)$ and $L_B(T,\theta)$. For any $\theta_1,\theta_2 \in \Theta$,

$$\begin{aligned} |L_I(T,\theta_1) - L_I(T,\theta_2)| &= \left|\tfrac{1}{T}\int_0^T [I(t,\theta_1) - I(t,\theta_2)]\,dt\right| \\ &\le \tfrac{1}{T}\int_0^T |I(t,\theta_1) - I(t,\theta_2)|\,dt \;\le\; T|\theta_1 - \theta_2|, \end{aligned} \tag{4.55}$$

where the second inequality is a consequence of Corollary 2. Similarly,

$$|L_B(T,\theta_1) - L_B(T,\theta_2)| \;\le\; T|\theta_1 - \theta_2|. \tag{4.56}$$

Equations (4.55) and (4.56) establish that Condition (b) of Fact 1 holds for both $L_I(T,\theta)$ and $L_B(T,\theta)$, thereby completing their proof of unbiasedness. ■

5 Discussion

This paper formulates Make-to-Stock (MTS) production inventory systems with backorders in stochastic fluid model (SFM) setting, and derives comprehensive formulas for IPA derivatives of the time averages of inventory and backorder levels with respect to the base-stock level and a production rate parameter. All IPA derivatives obtained are shown to be unbiased.

Since the theoretical results provide a basis for potential IPA-based design and on-line control of MTS systems, a number of issues merit additional discussion. First and foremost, how can IPA derivatives be used to good effect? The point of computing IPA derivatives with respect to control parameters is to extract sensitivity information on performance metrics in addition to the metrics themselves; in MTS systems, the metrics might be time averages of inventory levels and backorder levels, and the control parameters might be the base-stock level and production capacity. Certainly, long-run IPA derivatives may be used to optimize long-run performance metrics with respect to control parameters, using their IPA derivatives to search for optimal parameters [cf. Cassandras et al. (2002)]. Furthermore, in on-line control applications, one can continually measure performance metrics of interest, as well as their transient IPA derivatives, and devise control policies based on both that take actions at discrete times. In particular, one can employ a quick linear prediction (first-order Taylor Series) to predict system performance metrics under changed parameters, and use the prediction in the control policy. Finally, since a control action generally restarts the system from a new state, the transient IPA derivative would have to be reset and their computation restarted from the new (initial) state.

Second, it is worth emphasizing the role of right and left IPA derivatives. For example, suppose the control parameter is the base-stock level (i.e., $S(\theta) = \theta$) and the initial inventory level is precisely that same base-stock level (i.e., $I(0) = S(\theta)$), and consider changing the value of θ. Then, prospective raising of the base-stock level would make use of the right derivative, while prospective lowering of the base-stock level would make use of the left derivative. As those sided derivatives differ, both of them are necessary for on-line control applications.

Third, in regard to applications, it is further worth pointing out that the nonparametric nature of all IPA derivatives renders them usable in principle in both simulation and real-life systems. Of course, actual implementation of the corresponding formulas would require mapping IPA derivative estimators from the discrete-flow paradigm to the continuous-flow paradigm. As far as computer implementation is concerned, although the formulas appear to be complex, all IPA derivatives are readily computable, requiring modest computational resources (CPU time and storage). In fact, all IPA derivatives can be computed incrementally in time, with computational updates triggered by the occurrence of hitting times based on selected value changes in the state of the system (e.g., the inventory or backorder level becoming positive or ceasing to be positive).

Finally, a key issue is whether or not a fluid-flow paradigm can be successfully applied to production-inventory systems that have discrete rather than continuous flows. Since IPA derivatives for discrete-flow systems are biased, comparisons must be indirect and experiential. We mention that some prior experience with IPA-based predictions in the context of packet flows in telecommunications queueing systems supports such paradigm mix [see Cassandras et al. (2002)]. However, a real test of the success of an IPA-based application is the delivery of improvements in system performance metrics (e.g., service levels) over extant methods. This will be the subject of future work.

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References

J. Buzacott, S. Price, and J. Shanthikumar, "Service level in multi-stage MRP and base-stock controlled production systems," In G. Fandel, T. Gulledge, and A. Hones (eds.), *New Directions for Operations Research in Manufacturing*, pp. 445–463, Springer, Berlin, Germany, 1991.

C. G. Cassandras, Y. Wardi, B. Melamed, G. Sun, and C. G. Panayiotou, "Perturbation analysis for on-line control and optimization of stochastic fluid models," *IEEE Transactions on Automatic Control* vol. AC-47(8) pp. 1234–1248, 2002.

M. C. Fu, "Optimization via simulation: A review," *Annals of Operations Research* vol. 53 pp. 199–247, 1994a.

M. C. Fu, "Sample path derivatives for (s, S) inventory systems," *Operations Research* vol. 42 pp. 351–364, 1994b.

P. Glasserman, *Gradient Estimation via Perturbation Analysis*, Kluwer Academic Publishers, Boston, MA, 1991.

P. Glasserman, and S. Tayur, "Sensitivity analysis for base-stock levels in multiechelon production-inventory systems," *Management Science* vol. 41 pp. 263–281, 1995.

H. Heidelberger, X. R. Cao, M. Zazanis, and R. Suri, "Convergence properties of infinitesimal analysis estimates," *Management Science* vol. 34 pp. 1281–1302, 1988.

Y. C. Ho, and X. R. Cao, *Perturbation Analysis of Discrete Event Simulation*, Kluwer Academic Publishers, Boston, MA, 1991.

U. Karmarkar, "Lot sizes, lead-times and in-process inventories," *Management Science* vol. 33 pp. 409–419, 1987.

I. Paschalidis, Y. Liu, C. G. Cassandras, and C. Panayiotou, "Inventory Control for Supply Chains with Service Level Constraints: A Synergy between Large Deviations and Perturbation Analysis," *Annals of Operations Research* vol. 126 pp. 231–258, 2004.

R. Y. Rubinstein, and A. Shapiro, *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, John Wiley and Sons, New York, NY, 1993.

M. H. Veatch, "Using fluid solutions in dynamic scheduling," In S. B. Gershwin, Y. Dallery, C. T. Papadopoulos, and J. M. Smith (eds.), *Analysis and Modeling of Manufacturing Systems*, pp. 399–426, Kluwer, New York, 2002.

Y. Wardi, and B. Melamed, "Variational bounds and sensitivity analysis of continuous flow models," *Journal of Discrete Event Dynamic Systems* vol. 11(3) pp. 249–282, 2001.

Y. Wardi, B. Melamed, C. G. Cassandras, and C. G. Panayiotou, "On-line IPA gradient estimators in stochastic continuous fluid models," *Journal of Optimization Theory and Applications* vol. 115(2) pp. 369–405, 2002.

L. M. Wein, "Dynamic scheduling of a multiclass Make-to-Stock queue," *Operations Research* vol. 40 pp. 724–735, 1992.

Y. Zhao, and B. Melamed, "Make-to-Stock systems with backorders: IPA gradients," In Winter Simulation Conference (WSC 04), Washington D.C., December 5–8, pp. 559–567, 2004.

P. Zipkin, "Models for design and control of stochastic, multi-item batch production systems," *Operations Research* vol. 34 pp. 91–104, 1986.